

Measure Theory, 2008, Homework One

Some comments and remarks about the solutions

Q1: (2pts) Let (X, Σ) be a measure space (i.e. Σ is a σ -algebra on X). Assume Σ arises as the Borel sets resulting from some Polish topology on X . (Thus: (X, Σ) is in fact a standard Borel space). Let μ be a measure on (X, Σ) . Assume: (i) $\mu(X) = \infty$; (ii) μ is σ -finite; (iii) $\mu(\{x\}) = 0$ all $x \in X$ (that is to say, μ is atomless).

Show that we can write X as a disjoint union of sets

$$X = \dot{\bigcup}_{n \in \mathbb{N}} X_n$$

each with measure one.

This seemed to be the question giving the most trouble. Essentially the main problem is to get the pieces to come out exactly as measure one. The key lemma, after which all the other steps are routine, is this:

Lemma 0.1 *Let Y be a standard Borel space and let μ be a finite atomless Borel measure on Y . Then for all $c \in \mathbb{R}$ with*

$$0 \leq c \leq \nu(Y),$$

we can find $B \subset Y$ with $\mu(B) = c$.

Proof Using that Y is standard Borel, we can find an array of measurable sets

$$(B_s)_{s \in \mathbb{N}^{<\infty}}$$

such that: (i) $B_\emptyset = Y$; (ii) $B_s = \dot{\bigcup}_{n \in \mathbb{N}} B_{s \frown n}$ at each s ; (iii) and given $s, t \in \mathbb{N}^{<\infty}$ incompatible (i.e. there exists $\ell < \text{lh}(s), \text{lh}(t)$, $s(\ell) \neq t(\ell)$), B_s and B_t are disjoint; and finally, (iv) for each $f \in \mathbb{N}^{\mathbb{N}}$ there is at most one point in

$$\bigcup_{n \in \mathbb{N}} B_{f|_n}.$$

(We proved something like this in chapter 3...but it is easy to see directly.) Since the measure is atomless, for each $f \in \mathbb{N}^{\mathbb{N}}$ we have

$$\mu(B_{f|_n}) \rightarrow 0$$

as $n \rightarrow \infty$ by (iv). Moreover, since incompatible sequences are assigned disjoint sets, at each ℓ and each $\epsilon > 0$, there are only finitely many $s \in \mathbb{N}^\ell$ with $\mu(B_s) > \epsilon$.

Then we define measurable sets $C_0 = \emptyset \subset C_1 \subset C_2 \dots$ and finite sequences s_0, s_1, s_2, \dots such that at each n , $\mu(C_n \cup (B_{s_n} \setminus C_n)) < c$ and for any other $s \neq s_n$

$$\mu(C_n \cup (B_s \setminus C_n)) < c$$

implies $\mu(C_n \cup (B_s \setminus C_n)) \leq \mu(C_n \cup (B_{s_n} \setminus C_n))$.¹ Clearly the measures of the $B_{s_n} \setminus C_n$'s go to zero as $n \rightarrow \infty$ by finiteness of μ .

It remains to see we have $\mu(\bigcup_n (C_n)) = c$. But if not, we could choose some s with $\mu(\bigcup_n (C_n) \cup B_s) < c$. At all sufficiently large n we would have $\mu(B_{s_n}) < \mu(B_s)$, with a contradiction to the steps taken in our construction. □

¹In other words, we always choose s_n to be of maximal measure subject to $\mu(C_n \cup (B_{s_n} \setminus C_n)) < c$. Since the space has finite measure, (ii) (or (iii)) enables us to do this, as long as $\mu(C_n) < c$ (and if we ever reach $\mu(C_n) = c$ we stop, obviously).

Something like this argument is what I had in mind, though there are slightly different ways of handling this step.

A couple of people surprised me completely by pointing out an utterly different proof. They argued like this: After rescaling measure μ on Y , we can assume (Y, μ) is a standard Borel probability space. But then it will (by a quoted by never proved theorem in the notes) be isomorphic to the unit interval equipped with Lebesgue measure – and for this space the lemma is obvious.

Q2: (2pts) Let (X, Σ) be a measure space. Let $f : X \rightarrow \mathbb{R}$ be such that

$$f^{-1}[(-\infty, q)] \in \Sigma$$

all $q \in \mathbb{Q}$.

Show that f is measurable with respect to Σ (i.e. the pullback of any open set along f is in Σ).

People seemed to find this easy.

Q3: (2pts) Let X be the closed unit square, $[0, 1] \times [0, 1]$ equipped with the subspace topology (from the usual topology on \mathbb{R}^2). Let Σ be the resulting σ -algebra of Borel subsets of X . Let μ be Lebesgue measure on X (i.e. the restriction of the measure m on \mathbb{R}^2 defined on page one of the course notes).

Let

$$f : X \rightarrow \mathbb{R}$$

be defined by

$$(x, y) \mapsto x^2 y^2.$$

Let Σ_0 be the σ -algebra consisting of all sets of the form $A \times [0, 1]$ for $A \subset [0, 1]$ Borel.

Calculate $E(f|\Sigma_0)$, the conditional expectation of f with respect to Σ_0 .

This is easy, *once* the definitions are clear. In fact we have $g(x, y) = \frac{x^2}{3}$. (Note: It does not depend on the y coordinate.)

Q4: (4pts) Let

$$X = \prod_{n \in \mathbb{N}} \{0, 1\},$$

with the product topology. Let μ be the product measure on this space. (This is to say, for $A = \{f \in X : f(1) = \ell_1, f(2) = \ell_2, \dots, f(n) = \ell_n\}$, we have $\mu(A) = 2^{-n}$.)

For each finite $S \subset \mathbb{N}$ define

$$\begin{aligned} \psi_S : X &\rightarrow \mathbb{R} \\ f &\mapsto (-1)^{|\{n \in S : f(n) = 0\}|}. \end{aligned}$$

Show that $\{\psi_S : S \subset \mathbb{N}, S \text{ finite}\}$ gives an orthonormal basis for the Hilbert space $L^2(X, \mu)$.

People basically knew how to do this, though there was an issue on which some answers were less than clear: Showing $\int_X \psi_S d\mu = 0$ for $S \neq \emptyset$.

Here is the simplest way to do it: Choose some $n \in S$. Let $A_0 = \{f \in X : f(n) = 0\}$, $A_1 = \{f \in X : f(n) = 1\}$. Then $\mu(A_0) = \mu(A_1) = \frac{1}{2}$ and $X = A_0 \dot{\cup} A_1$. We can also define a measure preserving bijection $\Phi : A_0 \rightarrow A_1$ by $\Phi(f)(m) = f(m)$ for $m \neq n$ by $\Phi(f)(n) = 1$.

Since $\psi_S(\Phi(f)) = -\psi_S(f)$ we get $\int_{A_0} \psi_S d\mu = -\int_{A_1} \psi_S d\mu$, and so the integral over ψ_S on $X = A_0 \dot{\cup} A_1$ must come out as zero.