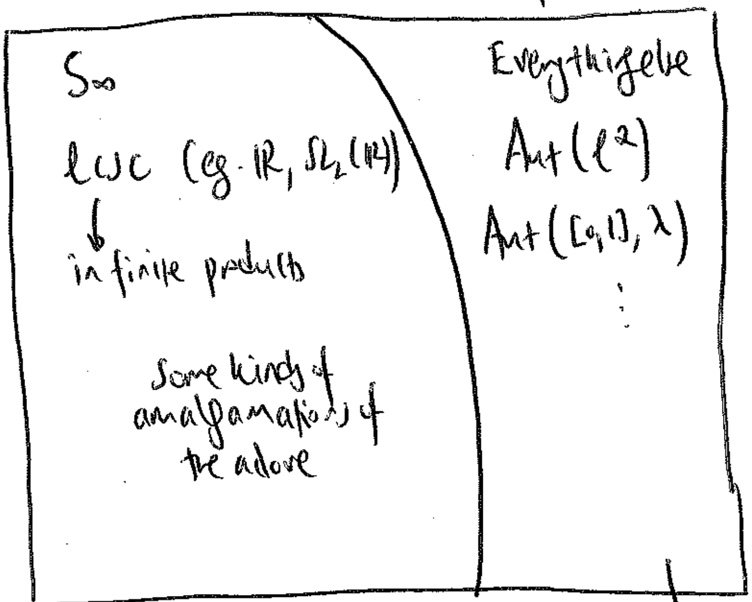
www.math.ucla.edu/~greg/fineorbits.pdf

Defn. S_∞ = permutations of $\mathbb{N}$ (topology of pointwise convergence)

Two general classes of Polish groups



Not understood.

But $Aut(\ell^2) \rightleftarrows \infty$ dimensional unitary group representations,

$Aut([0,1], \lambda) \rightleftarrows$ ergodic theory

The orbit equivalence relations are essentially understood.

S_∞: Scott analysis + Becker-Kechris on universality of the "logic action"

lcsc: Kechris: $E_G \le_B E_\infty$

Products of lcsc: Hjorth

The Scott Analysis

Version I:

let X be a Polish S_∞-space
(i.e. a separable completely metrizable space equipped with a continuous action of S_∞).

let $\{U_\ell \mid \ell \in \omega\}$ be a basis for X.

For $\vec{a}, \vec{b} \in \mathbb{N}^{<\omega}$, $V_{\vec{a},\vec{b}} = \{\sigma \in S_\infty \mid \sigma(\vec{a}) = \vec{b}\}$. (Note: these form an invariant basis for S_∞.)

For $x \in X$, $\varphi^x_{0,\vec{a},\vec{b}} = \{\ell \mid V_{\vec{a},\vec{b}} \cdot x \cap U_\ell\} \neq \phi$

$\varphi^x_{\alpha+1,\vec{a},\vec{b}} = \{ \varphi^x_{\alpha\, \vec{a}^\frown\vec{c}, \vec{b}^\frown\vec{d}} \mid \vec{c}, \vec{d} \text{ finite}\}$.

At λ a limit, $\varphi^x_{\lambda,\vec{a},\vec{b}} = \{ \varphi^x_{\alpha,\vec{a},\vec{b}} \mid \alpha < \lambda\}$.

Then: $\varphi^x_\alpha = \varphi^x_{\alpha,\phi,\phi}$

Fact: (i) $\exists$ (least) $\delta(x) < \omega_1$ s.t. $\forall \vec{a}, \vec{a}', \vec{b}, \vec{b}'$ $\left(\varphi^x_{\delta(x),\vec{a},\vec{b}} = \varphi^x_{\delta(x),\vec{a}',\vec{b}'} \text{ iff } \forall \alpha \in \omega_1 \left(\varphi^x_{\alpha,\vec{a},\vec{b}} = \varphi^x_{\alpha,\vec{a}',\vec{b}'}\right)\right)$.

(ii) $x \, E_{S_\infty} \, y \Rightarrow \delta(x) = \delta(y)$

(iii) $\varphi^x_{\delta(x)+2}$ is a complete invariant of $[x]_{S_\infty}$.

(iv) each $\varphi^x_\alpha \in HC$.

Version II:

let $\mathcal{L}$ be a countable language,

$\mathrm{Mod}(\mathcal{L})$ = $\mathcal{L}$-structures on $\mathbb{N}$, with basic open sets $\{\mathcal{M} \mid \mathcal{M} \models \varphi(\vec{a})\}$, φ a q.f. formula

$S_\infty \curvearrowright \mathrm{Mod}(\mathcal{L})$ by $\sigma \cdot \mathcal{M} \models \varphi(\vec{a})$ iff $\mathcal{M} \models \varphi(\sigma^{-1}(\vec{a}))$.

$\varphi^{\mathcal{M}}_{0,\vec{a}}(\vec{x})$ = q.f. type of $\vec{a}$,

$\varphi^{\mathcal{M}}_{\alpha+1,\vec{a}}(\vec{x}) = $ "$\{\varphi^{\mathcal{M}}_{\alpha,\vec{a}^\frown}(\vec{x},\vec{y}) : \vec{c} \in \mathcal{M}^{<\omega}\}$" $\left(= \bigwedge_{\vec{c} \in \mathcal{M}^{<\omega}} \exists \vec{y}\, \varphi^{\mathcal{M}}_{\alpha,\vec{a}^\frown\vec{c}}(\vec{x},\vec{y}) \wedge \forall \vec{y} \bigvee_{\vec{c} \in \mathcal{M}^{<\omega}} \varphi^{\mathcal{M}}_{\alpha,\vec{a}^\frown\vec{c}}(\vec{x},\vec{y})\right)$

$\varphi^{\mathcal{M}}_{\lambda,\vec{a}} = \bigwedge_{\alpha<\lambda} \varphi^{\mathcal{M}}_{\alpha,\vec{a}}$.

Version III: τ_0^x = original topology on X

$\tau_{\alpha+1}^x$ introduces basic open sets of the form
$\{y \in X \mid \overline{V_{\vec{a},\vec{b}}\cdot x}^{\tau_\alpha^x} = \overline{V_{\vec{c},\vec{d}}\cdot y}^{\tau_\alpha^x}\}$.

Theorem (essentially Scott, et al):

(i) $\varphi^x_{\delta(x)+2}$ is a complete invariant of $[x]_{S_\infty}$.
(ii) "$\varphi^x_\alpha = \varphi^y_\alpha$" is a $\Pi^0_{\alpha+n}$ equivalence rel^n. (n depending on α)
(iii) φ^x_α determines the invariant Π^0_α sets met by $[x]_{S_\infty}$.
(iv) "$\varphi^x_\alpha = \varphi^y_\alpha$" is "equivalent to" ($\leq_B$ in both directions) to an equivalence relation induced by an action of S_∞.

Applications (for S_∞)
(1) (Scott, ...) Each $[x]_{S_\infty}$ is a Borel subset of X ((i) + (iv)).
(2) If the orbits are bounded in Borel complexity, then E_{S_∞} is Borel
(if bounded by Π^0_α, E_{S_∞} bounded by $\Pi^0_{\alpha+\omega}$).

(3) There is an "effective" (eg in $L(\mathbb{R})$) map
$i: X/G \to \Pi^0_{\alpha+\omega}$
s.t. $i([x]_G) = i([y]_G) \Rightarrow [x]_G, [y]_G$ meet the same Π^0_α invariant sets
((ii) + (iii))

(4) (Friedman)
If E Borel, $E \leq_B E^X_{S_\infty}$ (arising from $S_\infty \curvearrowright X$ continuous),
then there is some other $E^{X'}_{S_\infty}$,
$E \leq_B E^{X'}_{S_\infty}$
where $E^{X'}_{S_\infty}$ is Borel.

. (uses "boundedness" + (iv)).

"Classical" Generalizations:

G a Polish group, X a Polish G-space.

$(1)^*$ (Ryll-Nardzewski) Each $[x]_G$ is Borel.

$(2)^*$ (Sami) $[x]_G$ bounded in complexity, then E_G Borel.

$(3)^*$? $(4)^*$??

Generalizing Scott

Let G be a Polish group, X a polish G-space.
Let d be a right invariant, compatible metric on G.
Let G_0 be a countable dense subgroup of G.

For $a \in \mathbb{Q}$, $V_a = \{g \in G \mid d(g,1) < a\}$.

For $x, y \in X$, $g, h \in G_0$, $a, b \in \mathbb{Q}^+$ set

$$(x, V_a \cdot g) \leq_0 (y, V_b \cdot h) \text{ if}$$

$$\overline{V_a \cdot g \cdot x} \subseteq \overline{V_b \cdot h \cdot y}.$$

$(x, V_a g) \leq_{\alpha+1} (y, V_b \cdot h)$ if $\forall\, V_{a'} \cdot g' \subseteq V_a \cdot g$, $\exists\, V_{b'} \cdot h' \subseteq V_b \cdot h$ s.t. $(x, V_{a'} \cdot g') \geq_\alpha (y, V_{b'} \cdot h')$

$\leq_\lambda$ (λ a limit) if at all $\alpha < \lambda$ we have $\leq_\alpha$.

Then: $x \simeq_\alpha y$ if $\forall V_a g$ $\exists V_b h, V_{a'} g'$

$$(x, V_{a'} g') \leq_\alpha (y, V_b h) \leq_\alpha (x, V_a g)$$

and vice versa ($x \rightleftarrows y$).

Then get $\delta(x)$ s.t. $(x, V_a g) \leq_{\delta(x)} (x, V_b h)$ and $q \in \mathbb{Q}^+$, $q < a, b$

$$\Rightarrow (x, V_{a-q}\, g) \leq_{\delta(x)+1} (x, V_{b+q}\, h).$$

Then $[x]_{\simeq_{\delta(x)+2}}$ determines $[x]_G$.

Gives an ordinal analysis proof of $(1)^*$, the sharp version of $(2)^*$ ($\alpha \to \alpha+\omega$ bound), and $(3)^*$.

We obtain analogs of (i) - (iii) for the Scott analysis, but not (iv).

Thm. (iv) fails for $G = \mathrm{Aut}(N_{\text{knight}}) \ltimes \prod_{\mathbb{N}} \mathrm{Aut}(N_{\text{knight}})$

where N_{knight} has

(a) a model of size $\aleph_1$ for its Scott sentence,

(b) if $\varphi \in L_{\omega_1\omega}$ and in some generic extension φ is the Scott sentence of an expansion of N_{knight}, then $|TC(\varphi)| \leq \aleph_1$.

Can define a weakening $\cong^*_\alpha$ of $\cong_\alpha$ (where $\cong_\alpha \subseteq \cong^*_\alpha \subseteq \cong^*_{\alpha+n}$, some finite n)

s.t. $E_1 \not\leq_B \cong^*_\alpha$.

Rmk: For G the universal Polish group (e.g. $G = \mathrm{Hom}([0,1]^{\mathbb{N}})$) the only known method for showing

$E \not\leq_B E_G$ (E Borel)

is to show $E_1 \leq_B E_G$.