Countable Borel equivalence relations, Borel reducibility, and orbit equivalence

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Contents

This is based on a sequence of four lectures given at Kobe during the 10th Asian Logic Conference.

I will discuss a little of the theory of Borel equivalence relations, from the point of view of Borel reducibility, and then try to indicate why countable Borel equivalence relations forced set theorists in this area to become acquainted with the research in *orbit equivalence*. This is an eclectic and tangly area which falls outside the usual neat divisions into the various clubs and sub-specialties which dominate so much of modern pure mathematics. Among other areas, it makes connections with functional analysis, geometric group theory, Lie groups and *super rigidity* in the sense of Margulis and Zimmer, operator algebras, ergodic theory, and only most recently descriptive set theory.

There are several peculiarities of this exposition. Let me state them explicitly, just so it is clear the extent to which they are intentional.

First of all, I am explicitly writing from the view point of a set theorist. Calculations which are routine given a perfect working knowledge of say the first eighteen chapters of [16] are omitted – thus the various assertions that certain functions are Borel or measurable or that certain spaces can naturally be viewed as standard Borel are all passed over without serious discussion or proof.

On the other hand, many well know results from outside set theory, in some cases quite minor, are often addressed. Although most of the proofs are at the level of sketches, alerting the reader to key ideas without unravelling all the details, I have tried to provide at least some indications of all the main ideas. The only real black box which is being used below is the *spectral theorem* for infinite dimensional unitary operators – in one place, in the discussion of *property (T)*, we will not only use the theorem but take certain steps which can only be justified after considerable reflection on the theorem and how it can it can be used in the context of group representations.

Many numerical calculations are skipped. Instead of fixing a $\epsilon > 0$ and writing $\|f - g\| < \epsilon$, I will just note $f \sim g$, leaving it as understood that with sufficient care we could go back and find the precise ϵ which would make the various inequalities unfold in the way we desire.

It should be understood that these notes travel through a lot of very deep work quickly, with all sorts of simplifications. Very often I am proving a theorem in a special case. Indeed very often I am taking one tiny bite from a much larger body of work and illustrating a key idea. The exposition here *does not* do justice to the very deep theory which has been developed in the area by mathematicians whose theorems I discuss below.

This is part of an introductory sequence of lectures. If you like, it is an "invitation". It is neither a text book nor a survey paper.

In these notes nothing at all is said about about the motivation for studying equivalence relations or why this has become a fashionable area in descriptive set theory. There are already a number of survey and expository papers, such as [9], [17], and [26], which do that job.

Finally, I should make it clear that not every theorem is attributed. In some cases this is laziness on my behalf, but more often because the results are folklore or have multiple and hazy origins. Only the theorems which I explicitly attribute to myself are due to me.

1 Lecture I

1.1 Borel equivalence relations

Definition X is a *Polish space* if it is separable and it accepts a complete metric. The *Borel sets* are those appearing in the σ-algebra generated by the open sets.

Examples (i) $\mathbb{R}$

(ii) $C([0,1])$, the collection of continuous functions from the unit interval to the reals, with distance

$$d(f,g) = \sup_{z\in[0,1]}|f(z)-g(z)|$$

inducing the topology.

(iii) For S a countable set, $\mathcal{P}(S)$, the collection of all subsets is a Polish space, under the natural identification of this space with

$$2^S = \{0,1\}^S = \prod_{\mathbb{N}}\{0,1\},$$

with product topology.

More generally, the countable product of Polish spaces is Polish in the product topology.

(iv) Any separable Banach space – such as ℓ^1, ℓ^2, c_0.

(v) The isometry group of a separable, complete metric space is Polish – take a dense subset of the space, $\{x_n : n \in \mathbb{N}\}$ and let

$$d(\phi,\psi) = \sum_{n\in\mathbb{N}} \min(2^{-n}, d(\phi(x_n), \psi(x_n)).$$

Viewing the Hilbert space ℓ^2 as a metric space in the norm metric, we obtain that the metric isomorphisms which are *linear* on ℓ^2 form a closed subspace of the group of isometries. This group is sometimes denoted as

$$U(\ell^2)$$

and referred to as the *unitary group* on ℓ^2.

Note that in general a closed subspace of a Polish space is again Polish

(vi) $M([0,1],\lambda)$, the measurable subsets of the unit interval, with the identification of sets agreeing almost everywhere, forms a Polish space. Its isometry group is alternately denoted by

$$\mathrm{Aut}([0,1],\lambda)$$

or

$$M_\infty([0,1],\lambda),$$

and consists of the measure preserving bijections.

Definition A set X equipped with a σ-algebra is said to be a *standard Borel space* if there is some choice of a Polish topology on X which gives rise to that σ-algebra as the collection of Borel subsets of X. A function between two standard Borel spaces,

$$f : X \to Y,$$

is said to be *Borel* if for any Borel $B \subset Y$ we have $f^{-1}[B]$ Borel in X.

The next two theorems are classical and can be found in [16].

Theorem 1.1 *A Borel subset A of a Polish space X is a standard Borel space – with the σ-algebra of all $B \subset A$ which are Borel in X.*

Theorem 1.2 *Any standard Borel space has cardinality one of:*

$$0, 1, 2, 3, ..., \aleph_0, 2^{\aleph_0}.$$

Moreover, any two standard Borel spaces of the same cardinality are Borel isomorphic.

The first part of this theorem holds for the cardinality of the number of equivalence classes of a Borel equivalence relation, as was shown in [24].

Definition An equivalence relation E on a standard Borel space X is said to be *Borel* if it is in the σ-algebra generated by the Borel rectangles, $A \times B$, where A and B are Borel subsets of X.

Theorem 1.3 *(Silver, [24]) If an equivalence relation E is Borel and it has uncountably many equivalence classes, then it has $2^{\aleph_0}$ many classes.*

However the *moreover* part of the classical theorem 1.2 fails. Among the Borel equivalence relations with uncountably many classes there is no simple or natural catalogue, and under any reasonable notion of isomorphism there are many, many non-isomorphic Borel equivalence relations with uncountably many classes.

1.2 Countable Borel equivalence relations

Definition A Borel equivalence relation is said to be *countable* if all its equivalence classes are countable.

Examples (i) $\text{id}(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 : x = y\}$. More generally, given a Polish space X we can form $\text{id}(X)$, the identity relation on X.

(ii) E_0, eventual agreement on infinite binary sequences – thus for $x, y \in 2^{\mathbb{N}}$ we set xE_0y if there is an N such that for all $n > N$

$$x(n) = y(n).$$

(iii) Given a countable group Γ, we can let it act on 2^Γ, the set of all functions $f : \Gamma \to \{0, 1\}$ in the product topological and measure theoretic structure, by

$$\gamma \cdot f(\sigma) = f(\gamma^{-1}\sigma),$$

and form the induced orbit equivalence relation E_Γ. This is sometimes called the *shift* action.

More generally, if we have any Borel action of a countable group Γ on a Polish space, we can form the induced orbit equivalence relation E_Γ.

Definition For E, F Borel equivalence relations on X, Y, we say that E *is Borel reducible to* F,

$$E \leq_B F,$$

if there is a Borel function $\theta : X \to Y$ such that for all $x_1, x_2 \in X$

$$x_1Ex_2 \Leftrightarrow \theta(x_1)F\theta(x_2).$$

By the classical isomorphism theorem for uncountable standard Borel spaces, whenever X, Y are uncountable we have $\text{id}(X) \leq_B \text{id}(Y)$.

Theorem 1.4 *E_0 is not Borel reducible to id$(\mathbb{R})$.*

Proof Let $\mathbb{Z}(2) = \{0, 1\}$ with addition mod 2. Let Γ be the group

$$\mathbb{Z}(2)^{<\infty},$$

consisting of all infinite sequences from $\mathbb{Z}(2)$ which are eventually zero. Let Γ act on $2^{\mathbb{N}}$ by

$$\gamma \cdot f(n) = \gamma(n) + f(n) \bmod 2,$$

and note that $E_\Gamma = E_0$.

Suppose $\theta : 2^{\mathbb{N}} \to \mathbb{R}$ is Borel with

$$x_1 E_0 x_2 \Rightarrow \theta(x_1) = \theta(x_2).$$

Let C be a comeager set on which θ is continuous. Take

$$C^* = \bigcap_{\gamma \in \Gamma} \gamma \cdot C$$

to obtain an invariant comeager set on which it is continuous.

For any $x \in C^*$ we will have that $[x]_{E_0}$, the equivalence class of x, is dense in C^*. Since θ is constant on this dense set, θ is constant on C^*. But C^* must be uncountable, so we can find an inequivalent $y \in C^*$ with $\theta(y) = \theta(x)$. □

Definition An equivalence relation E is *smooth* if $E \leq id(X)$ for some standard Borel space X.

Thus we have seen in 1.4 that E_0 is not smooth. Generalizations of the argument can be used to obtain corollaries of the proof. For instance:

Theorem 1.5 *Let Γ be a countable group acting by homeomorphisms on an uncountable Polish space X. If every orbit is dense, then E_Γ is not smooth.*

Now it gets hard.

Definition An equivalence relation E is *hyperfinite* if there is a increasing sequence of equivalence relations $E_1 \subset E_2 \subset E_3 ...$ such that each E_n is Borel with finite equivalence classes and

$$E = \bigcup_{n \in \mathbb{N}} E_n.$$

As shown in [15], if E is a countable Borel equivalence relation, then $E \leq_B E_0$ if and only if it is hyperfinite.

Theorem 1.6 *(Sullivan, Weiss, Wright) Let X be a Polish space and E a countable Borel equivalence relation on X. Then there is a comeager set $C \subset X$ on which E is hyperfinite.*

The original reference for this is [25], though a cleaner exposition is presented in [18]. It in effect tells us that beyond the first simplest step, categoricity arguments, working on comeager sets, will be of no use.

All the known proofs to obtain countable Borel equivalence relations beyond E_0 use measure theory.

1.3 Amenability and hyperfiniteness

Lemma 1.7 *Let $\mathbb{Z}$ act by Borel automorphisms on a standard Borel space X. Then the resulting orbit equivalence relation is hyperfinite.*

Proof For conceptual simplicity I will assume that X is Polish and the action is continuous.

Now if we are really, really lucky, the resulting orbits, $[x] = \{T^\ell(x) : \ell \in \mathbb{Z}\}$, will all be dense and so will each of the *forward orbits*,

$$\{T^\ell(x) : \ell \geq 0\},$$

and the *backwards orbits*,

$$\{T^\ell(x) : \ell \leq 0\}.$$

Now choose a sequence of decreasing non-empty open sets, $U_0 \supset U_1 \supset U_2 ...$ such that

$$\bigcap_{n\in\mathbb{N}} U_n = \emptyset.$$

We then set $xE_nT^\ell(x)$ if either

(i) $\ell = 0$; or

(ii) $\ell > 0$ and $x, T(x), T^2(x), ...T^\ell(x)$ are all *outside* U_n; or

(ii) $\ell < 0$ and $x, T^{-1}(x), T^{-2}(x), ...T^\ell(x)$ are all *outside* U_n.

A more complicated case is when the orbits are dense, but not necessarily in both directions. Then we choose for each x some basic open W_x so that $[x]$ has a last or first moment when it meets that open set. With some care we can do this so that $xEy \Rightarrow W_x = W_y$ and if we let $s(x)$ be that special last or first point, then

$$x \mapsto s(x)$$

is not only $\mathbb{Z}$-invariant but Borel.[1] We then let xE_ky if $x = y$ or for some $i, j \in \{-k, -k+1, ..., 0, 1, ...k\}$ we have $T^j(x) = s(x)$, and

$$T^i(y) = s(x) = s(y).$$

In general there is no guarantee, of course, that the orbits will be dense. The argument in this more typical case involves decomposing X into Borel subsets on which all orbits have the same closure and working on each of these components separately. Suffice to say there are technicalities, but they are not deep. □

Definition $\mathbb{F}_2 = \langle a, b\rangle$ is the free group on generators a and b. An element of $\mathbb{F}_2$ consists of *reduced* words in the letters $\{a, a^{-1}, b, b^{-1}\}$ – where *reduced* means there should be no adjacent appearances of a and a^{-1} or b and b^{-1}. We multiply elements of $\mathbb{F}_2$ by concatenating and then reducing.

For instance let $\sigma = ab^2a^{-1}$ and $\tau = ab^3aba^{-1}$. (The usual notational shortcut: I write ab^2a^{-1} instead of $abba^{-1}$.) We multiply by

$$\sigma\tau = ab^5aba^{-1}.$$

Technically speaking, the identity of $\mathbb{F}_2$ is the empty string. We usually denote this by e – as against lyrically just leaving an empty space and hoping it is recognized for its role as the identity of the group.

[1]Here is another way of looking at the argument: We obtain an invariant Borel set on which $E_\mathbb{Z}$ is smooth; the remainder of the proof consists in verifying that if a countable Borel equivalence relation is smooth, then it is hyperfinite.

Definition For Γ a countable group, we let $\ell^1(\Gamma)$ be the space of functions

$$f : \Gamma \to \mathbb{R}$$

with

$$\sum_{\sigma \in \Gamma} |f(\sigma)| < \infty.$$

For $f \in \ell^1(\Gamma)$ we let

$$\|f\| = \sum_{\sigma \in \Gamma} |f(\sigma)|.$$

We let Γ act on $\ell^1(\Gamma)$ by

$$\sigma \cdot f(\tau) = f(\sigma^{-1}\tau).$$

We then say that the group Γ is *amenable* if for any finite $F \subset \Gamma$ and $\epsilon > 0$ there is some $f \in \ell^1(\Gamma)$ with

$$\|f\| = 1$$

and

$$\|f - \sigma \cdot f\| < \epsilon$$

all $\sigma \in F$.

The choice taken here for defining amenability, in terms of *almost invariant vectors* in $\ell^1(\Gamma)$, is technical but convenient to our goals. A discussion of other characterizations and more can be found in [8], but it is worth giving an alternative characterization which is combinatorially more transparent and probably more appealing to a set theorist.

Lemma 1.8 *A countable group Γ is amenable if and only if there are "almost invariant finite subsets of Γ". That is to say, for all $\epsilon > 0, F \subset \Gamma$ finite, there is some finite non-empty $A \subset \Gamma$ such that for all $\gamma \in F$*

$$\frac{|A \Delta \gamma A|}{|A|} < \epsilon.$$

Proof Just in regards to notation, $|A|$ is the size of the set A and γA is the set $\{\gamma\sigma : \sigma \in A\}$ and $A\Delta\gamma A$ is the symmetric difference of the sets – the set of points in one but not the other.

The main issue in the proof is to go from the almost invariant functions to the almost invariant sets. So let us suppose $F \subset \Gamma, \epsilon > 0$ are given, and we have $f \in \ell^1(\Gamma)$ with

$$\|f - \gamma \cdot f\| < \frac{\epsilon}{|F|}\|f\|$$

all $\gamma \in F$. Without loss, $f \geq 0$.

At each $r \in \mathbb{R}$, let f_r be the characteristic function for the set on which f assumes value greater than r – that is to say, $f_r(h) = 1$ if $f(h) \geq r$, and $= 0$ otherwise.

Then at all $h \in \Gamma$

$$f(h) = \int_{\mathbb{R}^{\geq 0}} f_r(h) d\lambda(r).$$

Putting this into the inequality

$$\sum_{\gamma \in F} \|\gamma \cdot f - f\| < \epsilon \|f\|$$

we obtain

$$\sum_{\gamma \in F} \sum_{h \in \Gamma} \int_{\mathbb{R}^{\geq 0}} |\gamma \cdot f_r(h) - f_r(h)| d\lambda(r) < \epsilon \sum_{h \in \Gamma} \int_{\mathbb{R}^{\geq 0}} f_r(h) d\lambda(r).$$

Interchanging the order of summation, a legitimate step given that all the quantities involved are ≥ 0, we obtain

$$\int_{\mathbb{R}^{\geq 0}} \sum_{\gamma \in F} \sum_{h \in \Gamma} |\gamma \cdot f_r(h) - f_r(h)| d\lambda(r) < \epsilon \int_{\mathbb{R}^{\geq 0}} \sum_{h \in \Gamma} |f_r(h)| d\lambda(r).$$

Thus in particular there must be some $r > 0$ with

$$\sum_{\gamma \in F} \sum_{h \in \Gamma} |\gamma \cdot f_r(h) - f_r(h)| < \epsilon \sum_{h \in \Gamma} |f_r(h)|.$$

Take A to be the set of points at which $f(h) \geq r$. □

Lemma 1.9 $\mathbb{F}_2$ *is not amenable.*

Proof For $u \in \{a, a^{-1}, b, b^{-1}\}$ we let A_u be the reduced words beginning with u. For $f \in \ell^1(\mathbb{F}_2)$ we define f_u by

$$f_u(\sigma) = f(\sigma)$$

if $\sigma \in A_u$, and $f_u(\sigma) = 0$ otherwise.

Thus f is the sum of $f_a, f_{a^{-1}}, f_b, f_{b^{-1}}$, as well as its value on e. Note moreover that if $w \neq u^{-1}$, $w, u \in \{a, a^{-1}, b, b^{-1}\}$, then

$$w \cdot A_u \subset A_w.$$

We will take as our F the set $\{a, a^{-1}, b, b^{-1}\}$ and as our ϵ the value $\frac{1}{4}$. Let $f \in \ell_1(\Gamma)$ with $\|f\| = 1$. We will show there is some $\sigma \in F$ with

$$\|f - \sigma \cdot f\| \geq \frac{1}{4}.$$

First choose $u \in F$ with

$$\|f_u + f_{u^{-1}}\| \leq \frac{1}{2}.$$

Note that this implies

$$\|f_u\| + \|f_{u^{-1}}\| = \|f_u + f_{u^{-1}}\| \leq \frac{1}{2}.$$

Thus either $\|f_u\| \leq 1/4$ or $\|f_{u^{-1}}\| \leq 1/4$. For simplicity assume $\|f_{u^{-1}}\| \leq 1/4$. For $\sigma \notin A_{u^{-1}}$ we have $u \cdot \sigma \in A_u$ and $(u \cdot f)(u \cdot \sigma) = f(\sigma)$. This yields

$$\|(u \cdot f)_u\| \geq \|f - f_{u^{-1}}\| \geq \|f\| - \|f_{u^{-1}}\| \geq \frac{3}{4}.$$

Thus

$$\|u \cdot f - f\| \geq \|(u \cdot f)_u - f_u\| \geq \|(u \cdot f)_u\| - \|f_u\| \geq \frac{1}{4}.$$

□

Lemma 1.10 $\mathbb{Z}$ *is amenable.*

Proof Let

$$A_n = \{-n, -n+1, ..., 0, 1, ..., n-1, n\}.$$

For $\ell \in Z$ and $A \subset \mathbb{Z}$ we let $\ell \cdot A = \{\ell + k : k \in A\}$. It is then easily seen that

$$\frac{|A_n \Delta \ell \cdot A_n|}{|A_n|} \to 0$$

as $n \to \infty$. Thus if we let

$$f_n = \frac{1}{|A_n|}\chi_{A_n}$$

then each $\|f_n\| = 1$ and

$$\|f_n - \ell \cdot f_n\| \to 0$$

as $n \to \infty$. Thus given any finite $F \subset \mathbb{Z}$ and $\epsilon > 0$ we have

$$\|f_n - \ell \cdot f_n\| < \epsilon$$

all $\ell \in F$ and n sufficiently large. □

Definition A measure space (X, μ) is *standard* if X is a standard Borel space and μ is a σ-finite, atomless, Borel measure. It is *finite* if $\mu(X) < \infty$ and it is a *standard Borel probability* space if in addition $\mu(X) = 1$.

Any standard Borel probability space is measurable isomorphic to $([0,1], \lambda)$, the unit interval equipped with Lebesgue measure.

Theorem 1.11 *Let $\mathbb{F}_2$ act freely and by measure preserving Borel automorphisms on a standard Borel probability space (X, μ). Then the resulting orbit equivalence relation*

$$E_{\mathbb{F}_2} = \{(x, \sigma \cdot x) : \sigma \in \mathbb{F}_2\}$$

is not hyperfinite.

Proof Suppose instead for a contradiction

$$E_{\mathbb{F}_2} = \bigcup_{i \in \mathbb{N}} E_i$$

where the E_i's are finite Borel equivalence relations with $E_i \subset E_{i+1}$. Then at each $i \in \mathbb{N}, x \in X$ we let

$$f_{i,x}(\sigma) = \frac{1}{|[x]_{E_i}|}$$

if $xE_i\sigma^{-1} \cdot x$, and $= 0$ otherwise. (Here $|[x]_{E_i}|$ denotes the number of points E_i-equivalent to x.) Let

$$f_i(\sigma) = \int f_{i,x}(\sigma) d\mu(x).$$

Since the action is free we obtain each

$$f_{i,x} \in \ell^1(\mathbb{F}_2)$$

with $\|f_{i,x}\| = 1$. Interchanging integration with summation yields

$$\|f_i\| = \sum_{\sigma \in \mathbb{F}_2} \int f_{i,x}(\sigma) d\mu(x)$$

$$= \int \sum_{\sigma \in \mathbb{F}_2} f_{i,x}(\sigma) d\mu(x)$$

$$= \int \|f_{i,x}\| d\mu(x) = 1.$$

Claim: For each $\sigma \in \mathbb{F}_2$

$$\lim_{i\to\infty} \|f_i - \sigma \cdot f_i\| \to 0.$$

Proof of Claim: Let $A_i = \{x : xE_i\sigma \cdot x\}$. Since $\bigcup_{i\in\mathbb{N}} E_i = E_{\mathbb{F}_2}$ we obtain

$$\mu(A_i) \to 1$$

as $i \to \infty$. Moreover if $x \in A_i$ then for any $\tau \in \mathbb{F}_2$

$$\tau^{-1} \cdot xE_i x \Leftrightarrow \tau^{-1} \cdot xE_i \sigma \cdot x$$

$$\Leftrightarrow (\sigma\tau)^{-1} \cdot (\sigma \cdot x) E_i \sigma \cdot x,$$

and thus

$$f_{i,\sigma\cdot x}(\sigma\tau) = f_{i,x}(\tau);$$

equivalently we can state this as saying that for any $\tau \in \mathbb{F}_2$

$$f_{i,\sigma\cdot x}(\tau) = f_{i,x}(\sigma^{-1}\tau).$$

Thus

$$\int_{A_i} \sigma \cdot f_{i,x}(\tau) d\mu = \int_{A_i} f_{i,x}(\sigma^{-1}\tau) d\mu$$

$$= \int_{A_i} f_{i,\sigma\cdot x}(\tau) d\mu$$

$$= \int_{\sigma\cdot A_i} f_{i,x}(\tau) d\mu,$$

since σ acts in a measure preserving manner. Thus

$$\|f_i - \sigma \cdot f_i\| = \sum_{\tau\in\mathbb{F}_2} (|\int f_{i,x}(\tau) d\mu - \int f_{i,x}(\sigma^{-1}\tau) d\mu|)$$

$$\leq \sum_{\tau\in\mathbb{F}_2} (|\int_{\sigma\cdot A_i} f_{i,x}(\tau) d\mu - \int_{A_i} f_{i,x}(\sigma^{-1}\tau) d\mu|) + \sum_{\tau\in\mathbb{F}_2} (\int_{X\setminus\sigma\cdot A_i} f_{i,x}(\tau) d\mu + \int_{X\setminus A_i} f_{i,x}(\sigma^{-1}\tau) d\mu)$$

$$= \int_{X\setminus\sigma\cdot A_i} \sum_{\tau\in\mathbb{F}_2} f_{i,x}(\tau) d\mu + \int_{X\setminus A_i} \sum_{\tau\in\mathbb{F}_2} f_{i,x}(\sigma^{-1}\tau) d\mu$$

$$= \mu(X \setminus \sigma \cdot A_i) + \mu(X \setminus A_i).$$

Since σ acts in a measure preserving manner, this in turn equals

$$2\mu(X \setminus A_i),$$

which goes to 0 as $i \to \infty$. (Claim□)

But now for any finite $F \subset \mathbb{F}_2$ and $\epsilon > 0$ we will have at all sufficiently large i

$$\forall \sigma \in F(\|f_i - \sigma \cdot f_i\| < \epsilon),$$

with a contradiction to non-amenability of $\mathbb{F}_2$. □

All we used about $\mathbb{F}_2$ is its non-amenability. Thus the proof shows:

Theorem 1.12 *Let Γ be a non-amenable group acting freely and by measure preserving transformations on a standard Borel probability space (X, μ). Then the resulting orbit equivalence relation is not hyperfinite.*

As a corollary to this theorem we obtain another proof that $\mathbb{Z}$ is amenable.

Theorem 1.13 *(Gao-Jackson) Let Γ be a countable abelian group acting by Borel automorphisms on a standard Borel probability space X. Then the resulting orbit equivalence relation E_Γ is hyperfinite.*

Their long proof of this deep theorem is still yet to be published.

While we do not know whether this theorem can be extended to amenable groups, in the measure theoretic context the situation is settled.

Theorem 1.14 *(Ornstein-Weiss, [2], [22]) If Γ is an amenable group acting by measure preserving transformations on a standard Borel probability space (X, μ), then there is a conull set $M \subset X$ with*

$$E_\Gamma|_M$$

hyperfinite.

2 Lecture II

2.1 Orbit equivalence

Since all the known proofs to distinguish countable Borel equivalence relations $\geq_B E_0$ rely on measure theory, one is led to consider a definition which predates any of the work by descriptive set theorists on equivalence relations and arises by considering an equivalence relation up to its measure theoretic properties.

Definition An equivalence relation E on a standard measure space (X, μ) is *measure preserving* if for any two measurable $A, B \subset X$, and Borel bijection

$$\phi : A \to B,$$

with $\phi(x)Ex$ all $x \in A$, we have

$$\mu(A) = \mu(B).$$

(This arises for instance if E is induced by the measure preserving action of a countable group – for any such $\phi : A \to B$ we can decompose A into pieces $(A_\gamma)_{\gamma\in\Gamma}$ with $\phi(x) = \gamma \cdot x$ all $x \in A_\gamma$; then ϕ will be measure preserving on each A_γ, and hence on A.)

E is *ergodic* if for any invariant, Borel A we have that A is either null or conull. (Here *invariant* means that if $x \in A$ and yEx, then as well we must have $y \in A$).

Two ergodic, measure preserving, countable, Borel equivalence relations E_1, E_2 on standard Borel measure spaces $(X_1, \mu_1), (X_2, \mu_2)$ are *orbit equivalent* if there is a measure preserving, Borel bijection

$$\phi : X_1 \to X_2$$

such that for μ_1-a.e. $x \in X_1$ we have exact preservation of its orbit under ϕ – namely,

$$\{\phi(y) : yE_1x\} = \{z : zE_2\phi(x)\}.$$

Traditionally orbit equivalence has focused on the case of ergodic equivalence relation. An ergodic equivalence relation should be thought of as irreducible – we cannot divide the space up into two non-trivial subspaces which are E-unrelated. An analogue of *simple group* or *indecomposable module.*

Building on earlier results in the area, the following was proved in early 2002.

Theorem 2.1 *(Hjorth, [10]) Any countable non-amenable group has at least two free, ergodic, measure preserving actions on standard Borel probability spaces up to orbit equivalence.*

I may have been happy with the result at the time, but it raised more questions than it answered. The proof showed for the class of *property (T)* groups we have continuum many actions up to orbit equivalence. Connes, Schmidt, and Weiss had previously shown that for non-amenable, non-(T) groups we have at least 2. Hence in general we have at least two, but no hint how to obtain larger numbers for groups such as $\mathbb{F}_2$.

We will start working towards a result of Gaboriau and Popa a couple of years later which showed the non-abelian free groups to have continuum many orbit inequivalent actions. First we need some entirely new concepts.

2.2 The spectral theorem and direct integrals of representations

The purpose of this section is to provide a very fast introduction to how the spectral theorem for abelian operators can be levered to analyze the representations of a group with a normal abelian subgroup. The key idea is to reduce to the case of irreducible representations.

In general given a finite dimensional unitary representation we can write it as a *direct sum* of irreducible representations. Given an invariant closed subspace, its orthogonal complement will be invariant as well, and we can decompose the space into two invariant orthogonal pieces. Finite dimensionality ensures the process will terminate after a finite number of steps.

The situation with infinite dimensional representations is more subtle. There is no guarantee of termination after finitely many steps, and indeed we can not generally write a unitary representation as a direct sum of irreducible representations. Rather one obtains that the representation can be written as a kind of *direct integral* of irreducible representations.

We will want to apply this kind of analysis to the case of a countable group Γ with a normal abelian subgroup Δ. I will try to describe the process informally, in loose general terms, before going on to set up the formal details.

Irreducible representations of abelian groups are one dimensional, and thus the irreducible representations of Δ correspond to elements of the dual group, $\hat{\Delta}$, consisting of homomorphisms from Δ to the circle group S^1 of complex numbers of norm one. In other words, if $\Delta \curvearrowright \mathcal{H}$ is an irreducible representation, then $\mathcal{H}$ will be one dimensional and there will be a homomorphism $\chi : \Delta \to S^1$ with $\delta \cdot v = \chi(\delta)v$ all $\delta \in \Delta$, $v \in \mathcal{H}$.

We can lever this in the situation that $\Gamma \to U(\mathcal{H})$ is a unitary representation. In very rough terms – and the details are cleaned up in the passages below – we can first of all write the unitary representation so that at the level of Δ it is a direct integral of irreducible Δ representations. After suitably rearranging the space notationally, we thereby obtain

$$\mathcal{H} = \int_{\hat{\Delta}} \mathcal{H}_\chi \nu(\chi),$$

where each $\mathcal{H}_\chi$ is a direct sum of one dimensional representations of Δ corresponding to the character χ and ν is a probability measure on the dual $\hat{\Delta}$. This in place, we can think of any element in $\mathcal{H}$ as being a ν-measurable function which assigns each to element $\chi \in \hat{\Delta}$ a corresponding $w_\chi \in \mathcal{H}_\chi$. In the notation above we are thinking of $w \in \mathcal{H}$ in exactly that way – as a function. The key point here is that the normality of Δ means that we can think of Γ/Δ as acting in a non-singular manner on the space $(\hat{\Delta}, \nu)$.

In the literature, this whole process, of analyzing a group representation in terms of a direct integral of irreducible representations for some normal abelian subgroup, is sometimes referred to as the "Mackey machine." The various mathematical assertions which I make in the text below are standard for the area. They can be found in [20] or [30].

Definition Let $\mathcal{H}$ be a separable Hilbert space. $B(\mathcal{H})$ is the collection linear operators

$$\phi : \mathcal{H} \to \mathcal{H}$$

which are bounded in the sense that there exists a single bound $c > 0$ such that for all $v \in \mathcal{H}$

$$\|\phi(v)\| \leq c\|v\|.$$

$U(\mathcal{H}) \subset B(\mathcal{H})$ is the collection of *unitary operators*; in other words, the collection of linear bijections $\phi : \mathcal{H} \to \mathcal{H}$ with

$$\langle \phi(v), \phi(w) \rangle = \langle v, w \rangle$$

for all $v, w \in \mathcal{H}$. For $\phi \in B(\mathcal{H})$ we let ϕ^*, the adjoint, be defined by

$$\langle \phi(v), w \rangle = \langle v, \phi^*(w) \rangle.$$

In the case that ϕ is unitary one has

$$\phi^{-1} = \phi^*.$$

Definition Let (X, μ) be a standard Borel probability space. Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be a collection of separable Hilbert spaces and let $(X_n)_{n\in\mathbb{N}}$ be a partition of X into measurable sets. We then define

$$\mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu)$$

to be the collection of all functions

$$f : X \to \bigcup_n \mathcal{H}_n$$

such that f is *measurable*, in the sense that for each $n \in \mathbb{N}$, $\epsilon > 0$, and $v \in \mathcal{H}_n$ the set

$$\{x \in X_n : \|f(x) - v\| < \epsilon\}$$

is measurable, and

$$\int_X \langle f(x), f(x)\rangle d\mu(x) < \infty,$$

and $f(x) \in \mathcal{H}_n$ all $x \in X_n$. We then view $\mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu)$ as a Hilbert space in its own right: For $f, g \in \mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu)$, $c \in \mathbb{C}$

1. $\|f\| = \int_X \langle f(x), f(x)\rangle d\mu(x)$
2. $(f + g)(x) = f(x) + g(x)$
3. $(cf)(x) = c(f(x))$
4. $\langle f, g\rangle = \int_X \langle f(x), g(x)\rangle d\mu(x)$.

Given $\phi \in B(\mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu))$ we say that ϕ is a *multiplication operator* if there is some $g : X \to \mathbb{C}$ with the property that for any $f \in \mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu)$

$$\phi(f)(x) = g(x) \cdot f(x)$$

a.e. $x \in X$.

Definition For Γ a countable group and $\mathcal{H}$ a Hilbert space, a *unitary representation* of Γ is a homomorphism

$$\varphi : \Gamma \to U(\mathcal{H}),$$

$$\gamma \mapsto \varphi_\gamma.$$

The representation is said to be *irreducible* if there is no closed non-trivial proper subspace

$$\mathcal{H}_0 \subset \mathcal{H}$$

which is invariant under all the operators $(\varphi_\gamma)_{\gamma\in\Gamma}$.

The only kinds of group representations we will want to consider are unitary representations. Thus we will usually just say *representation* and leave it understood that it is unitary representation.

With the definitions in hand, we can at least start the relevant form of the spectral theorem.

Theorem 2.2 *Let $\mathcal{H}$ be a separable Hilbert space. Let $A \subset B(\mathcal{H})$ be a subset which is commutative in the sense that*

$$\phi\psi(v) = \psi\phi(v)$$

all $v \in \mathcal{H}, \phi, \psi \in B(\mathcal{H})$ and closed under adjoints in the sense that $\phi^* \in A$ whenever $\phi \in A$. Then there is a standard Borel probability space (X, μ) and an isomorphism of Hilbert spaces

$$\pi : \mathcal{H} \cong \mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu)$$

such that for each $a \in A$ there exists measurable bounded $g_a : X \to \mathbb{C}$ such that

$$\pi(a \cdot v)$$

always equals the function

$$x \mapsto g_a(x)(\pi(v))(x).$$

In other words, every element of A is conjugate by π to a multiplication operator on $\mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu)$. Note that in particular this theorem implies that every irreducible representation of an abelian group is one dimensional – in other words, it corresponds to a character.

Definition Let $S^1 = \{\xi \in \mathbb{C} : |\xi| = 1\}$, equipped the group operation of complex multiplication. For Δ an abelian group, the *dual group* or *character group* of Δ, written $\hat{\Delta}$, consists of all homomorphisms

$$\chi : \Delta \to S^1.$$

Theorem 2.3 *Let Δ be a countable group. Let*

$$\varphi : \Delta \to U(\mathcal{H})$$
$$\delta \mapsto \varphi_\delta$$

be a unitary representation of Δ on a separable Hilbert space $\mathcal{H}$. Then we can find a standard Borel probability space (X, μ), along with $(\mathcal{H}_n)_{n\in\mathbb{N}}$ a collection of separable Hilbert spaces, and $(X_n)_{n\in\mathbb{N}}$ a partition of X into measurable sets, and an isomorphism

$$\pi : \mathcal{H} \cong \mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu),$$

and measurable assignments

$$\psi_\delta : X \to \bigcup_{n\in\mathbb{N}} U(\mathcal{H}_n)$$

for each $\delta \in \Delta$ such that:

1. *if $x \in X_n$ then $\psi_\delta(x) \in U(\mathcal{H}_n)$;*
2. *if $x \in X_n$ then as representation of Δ*

   $$\delta \mapsto \psi_\delta(x)$$

   is an irreducible representation;
3. *there is an isomorphism of Hilbert spaces*

   $$\pi : \mathcal{H} \cong \mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu)$$

   such that for any $\delta \in \Delta$ and $v \in \mathcal{H}$

   $$\pi(\varphi_\delta(v))(x) = \psi_\delta(x)(\pi(v)(x)).$$

The proof of this theorem derives from 2.2. In essence the idea is to look at all the operators

$$P \in B(\mathcal{H})$$

which are *projections* in the sense that $P^2 = P$ and commute with the representation, in the sense that for each $\delta \in \Delta$ and $v \in \mathcal{H}$

$$P(\varphi_\delta(v)) = \varphi_\delta(P(v)).$$

These can be thought of as measuring the failure of the representation to be irreducible, in the sense that if this collection of projections is trivial then the representation is already irreducible.

We then form a maximal abelian A in the collection of projections which commute with the representation φ. We apply 2.2 to write

$$\pi : \mathcal{H} \cong \mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu)$$

with each $P \in A$ conjugate to a multiplication operator. It follows from the P being a projection that at a.e. $x \in X$ we have

$$\pi(P(v))(x) \in \{\vec{0}, v(x)\}.$$

It is then a tedious, though routine, calculation to verify that at each $\delta \in \Delta$ there will be an assignment

$$\psi_\delta : X \to \bigcup_{n \in \mathbb{N}} U(\mathcal{H}_n)$$

such that at $v \in \mathcal{H}$

$$\pi(\varphi_\delta(v))(x) = \psi_\delta(x)(\pi(v)(x)).$$

Our interest in this decomposition of representations into direct integrals of irreducibles will be in particular directed to the case that the group Δ is a normal abelian subgroup of some larger group Γ which comes equipped with some representation

$$\varphi : \Gamma \to U(\mathcal{H}),$$

$$\gamma \mapsto \varphi_\gamma.$$

In this special case, we obtain the following:

Theorem 2.4 *Let Γ be a countable group and $\Delta \triangleleft \Gamma$ be a normal abelian subgroup. Let*

$$\varphi : \Gamma \to U(\mathcal{H})$$

$$\gamma \mapsto \varphi_\gamma$$

be a unitary representation of Γ on a separable Hilbert space $\mathcal{H}$. Then we can find a Borel probability measure μ on $\hat{\Delta}$ along with $(\mathcal{H}_n)_{n \in \mathbb{N}}$ a collection of separable Hilbert spaces, and $(X_n)_{n \in \mathbb{N}}$ a partition of $\hat{\Delta}$ into measurable sets, an isomorphism

$$\pi : \mathcal{H} \cong \mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu),$$

and a non-singular action of Γ on $(\hat{\Delta}, \mu)$,

$$a : \Gamma \times \hat{\Delta} \to \hat{\Delta},$$

such that for each $\delta \in \Delta$:

1. *there is an isomorphism of Hilbert spaces*

$$\pi : \mathcal{H} \cong \mathcal{H}((\mathcal{H}_n)_n, (X_n)_n, \mu)$$

such that for any $\delta \in \Delta$, $\chi \in \hat{\Delta}$, $v \in \mathcal{H}$

$$\pi(\varphi_\delta(v))(\chi) = \chi(\delta)(\pi(v)(\chi));$$

2. *for each* $\gamma \in \Gamma$, $\chi \in \hat{\Delta}$, $\delta \in \Delta$

$$a(\gamma, \chi)(\delta) = \chi(\gamma^{-1}\delta\gamma);$$

3. *for each* $\gamma \in \Gamma$, $v \in \mathcal{H}$, *and measurable* $B \subset \hat{\Delta}$

$$\int_B \|(\pi(v))(x)\| d\mu(x) = \int_{\gamma \cdot B} \|(\pi(\varphi_\gamma(v)))(x)\| d\mu.$$

There is more that can be said with regards to point 3, in fact more that probably should be said. First of all when I write $\gamma \cdot B$ in the subscript of the second integral, I mean this with respect to the indicated action a of Γ on $\hat{\Delta}$; more formally we should probably write $\{a(\gamma, x) : x \in B\}$ instead of $\gamma \cdot B$. There is also a kind of naturalness to the action of Γ on the dual space $\hat{\Delta}$, which arises from the induced action of Γ on the representations of Δ. This is discussed thoroughly in [20], and while the ideas are straightforward and very intuitive, the effort to make them precise notationally would take us too far afield. Indeed the statement at 3 is really a crude approximation to those ideas, justified only by its sufficiency in the course of the measure theoretic arguments underlying proof of relative property (T) in the next section.

2.3 Relative property (T)

Definition Let $\mathrm{SL}_2(\mathbb{Z})$ be the collection of matrices of the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with integer entries and determinant, $a_{11}a_{22} - a_{21}a_{12} = 1$.

This forms a group under matrix multiplication. The main issue is existence of inverses – but we can row reduce every element of $\mathrm{SL}_2(\mathbb{Z})$ down to the identity using elementary row operations and verify that each of those will have an inverse in $\mathrm{SL}_2(\mathbb{Z})$.

Definition Think of $\mathbb{Z}^2$ as the collection of 2×1 column matrices with addition conducted pointwise on each coordinate. Let $\mathrm{SL}_2(\mathbb{Z})$ act it in the obvious way.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} a_{11}m + a_{12}n \\ a_{21}m + a_{22}n \end{bmatrix}.$$

In this way each element of $\mathrm{SL}_2(\mathbb{Z})$ induces an automorphism of $\mathbb{Z}^2$, and we can form the semi-direct product

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$$

with the group operation

$$(M, \vec{a})(N, \vec{b}) = (MN, N^{-1} \cdot \vec{a} + \vec{b}).$$

Definition Let $\hat{\mathbb{Z}^2}$ be the *character group* of $\mathbb{Z}^2$ – that is to say the collection of all homomorphism

$$\chi : \mathbb{Z}^2 \to S^1,$$

where S^1 is the collection of complex numbers with norm one considered as a group under complex multiplication.

Facts 2.5 *(i) Every $\chi \in \hat{\mathbb{Z}^2}$ has the form*

$$\chi : \begin{bmatrix} m \\ n \end{bmatrix} \mapsto e^{2(sm+tn)\pi i},$$

for some $s, t \in \mathbb{R}$.

Definition I will write $\chi_{(s,t)}$ for the character

$$\chi_{(s,t)} : \begin{bmatrix} m \\ n \end{bmatrix} \mapsto e^{2(sm+tn)\pi i}.$$

Later on we will need to consider $\mathrm{SL}_2(\mathbb{Z})$ acting on these characters using its canonical action on $\mathbb{R}^2$ and hence $\mathbb{R}^2/\mathbb{Z}^2$. I will write $M \cdot (s,t)$ to indicate $(sa_{11} + ta_{12}, ta_{21} + ta_{22})$, where

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

I will use I to denote the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Facts 2.6 *(Continued) (ii) From (i), there is a natural isomorphism between $\hat{\mathbb{Z}^2}$ and $\mathbb{R}^2/\mathbb{Z}^2$.*

(iii) If $\mathbb{Z}^2$ acts in a unitary (i.e. a linear action with $\langle \vec{a} \cdot v, \vec{a} \cdot u \rangle$ is equal to $\langle \vec{v}, \vec{u} \rangle$ for all $u, v \in \mathcal{H}$, $\vec{a}, \vec{b} \in \mathbb{Z}^2$) manner on a one dimensional complex Hilbert space $\mathcal{H} = \mathbb{C} \cdot u$, then there are $s, t \in \mathbb{R}$ with

$$\begin{bmatrix} m \\ n \end{bmatrix} \cdot u = e^{2(sm+tn)\pi i} u.$$

Thus there is an exact correspondence between irreducible unitary representations of $\mathbb{Z}^2$ and elements of the dual group $\hat{\mathbb{Z}^2}$.

(iv) If $\mathcal{W}$ is a Hilbert space on which Γ acts unitarily, and $\mathcal{V}$ a $\mathbb{Z}^2$ invariant subspace, with

$$\begin{bmatrix} m \\ n \end{bmatrix} \cdot u = e^{2(sm+tn)\pi i} u,$$

all $u \in \mathcal{V}$. Then for any $u \in \mathcal{V}$, $\begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{Z}^2$ and

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we have

$$(M, \vec{0})(I, \begin{bmatrix} m \\ n \end{bmatrix})(M^{-1}, \vec{0}) = (I, M \cdot \begin{bmatrix} m \\ n \end{bmatrix}),$$

and hence for $v = (M^{-1}, \vec{0}) \cdot u$

$$(I, \begin{bmatrix} m \\ n \end{bmatrix}) \cdot v = (M^{-1}, \vec{0}) \cdot (I, M \cdot \begin{bmatrix} m \\ n \end{bmatrix})) \cdot u$$

$$= (M^{-1}, \vec{0}) \cdot e^{2(sma_{11}+sna_{12}+ta_{21}m+ta_{22}n)\pi i} u = e^{2(sma_{11}+sna_{12}+ta_{21}m+ta_{22}n)\pi i} v$$

$$= e^{2((a_{11}s+a_{21}t)m+(a_{12}s+a_{22}t)n)\pi i} v = \chi_{M^t(s,t)}(\begin{bmatrix} m \\ n \end{bmatrix})v.$$

Thus if $\mathbb{Z}^2$ *acts on* $\mathbb{C}u$ *as indicated by*

$$\chi_{(s,t)}$$

then it acts on $\mathbb{C}v = \mathbb{C}M^{-1} \cdot u$ *as indicated by*

$$\chi_{M^t(s,t)},$$

where M^t *is the transpose of* M.

Notation Since $\hat{\mathbb{Z}^2}$ is canonically isomorphic as a space to $\mathbb{R}^2/\mathbb{Z}^2$, we can take the induced $\mathrm{SL}_2(\mathbb{Z})$ invariant measure μ which arises from the usual Lebesgue measure on $\mathbb{R}^2/\mathbb{Z}^2$.

This is in fact equal to the *Haar measure* on $\hat{\mathbb{Z}^2}$ – the unique two sided invariant Borel probability measure which arises from viewing it as a compact group in point wise multiplication.

Theorem 2.7 *(Kazhdan)* $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ *has "relative property (T)" over* $\mathbb{Z}^2$ *– in other words, if*

$$\Gamma \to U(\mathcal{H})$$

is a unitary representation of Γ *on a Hilbert space* $\mathcal{H}$ *with "almost invariant vectors", then there is a "non-trivial"* $\mathbb{Z}^2$*-invariant vector.*

Definition Here when we say that a representation

$$\varphi : \Gamma \to U(\mathcal{H})$$

$$\gamma \mapsto \varphi_\gamma$$

has *almost invariant vectors* we mean that for all $\epsilon > 0$ and finite $F \subset \Gamma$ then we can find some $v \in \mathcal{H}, v \neq 0$ with

$$\|\varphi_\gamma(v) - v\| < \epsilon\|v\|$$

all $\gamma \in F$. Then $\mathbb{Z}^2 < \Gamma$ has a *non-trivial invariant vector* if there is some $v \in \mathcal{H}$ with $v \neq 0$ with

$$\varphi_{\vec{a}}(v) = v$$

all $\vec{a} \in \mathbb{Z}^2$.

Proof I will sketch a proof which was discovered independently by Burger and Shalom.

Let us start with some $v \in \mathcal{H}$ which has

$$\gamma \cdot v \sim v$$

all

$$\gamma \in \{(I, \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (I, \begin{bmatrix} 0 \\ 1 \end{bmatrix}), (\begin{bmatrix} 1 & \pm 3 \\ 0 & 1 \end{bmatrix}, \vec{0}), (\begin{bmatrix} 1 & 0 \\ \pm 3 & 1 \end{bmatrix}, \vec{0})\}.$$

Now for the non-trivial appeal to the spectral theorem. Appropriately understood, and applied to the abelian group $\mathbb{Z}^2$, this enables us to write the Hilbert space $\mathcal{H}$ as a direct integral

$$\mathcal{H} = \int_{\hat{\mathbb{Z}}^2} \mathcal{H}_\chi d\nu(\chi),$$

where ν is a Borel probability measure on $\hat{\mathbb{Z}}^2$, and where each $\mathcal{H}_\chi$ is a Hilbert space on which $\mathbb{Z}^2$ acts unitarily in accordance with χ – that is to say, for any $\vec{a} \in \mathbb{Z}^2$ and $w \in \mathcal{H}$

$$(I, \vec{a}) \cdot w = \int (\vec{a} \cdot w)_\chi d\nu(\chi) = \int \chi(\vec{a}) w_\chi d\nu(\chi).$$

Appealing to this decomposition, we can also write

$$v = \int_{\hat{\mathbb{Z}}^2} v_\chi d\nu(\chi).$$

Note then that by almost invariance of v we get that $\|v_\chi\|$ is very small for χ outside of some small neighborhood of $\vec{0}$ in $\mathbb{R}^2/\mathbb{Z}^2$. In fact, we may as well assume there is a tiny $\delta > 0$ such that

$$v_\chi = 0$$

unless

$$\chi = \chi_{(s,t)}$$

for some $s, t \in \mathbb{R}, |s|, |t| < \delta$. This is setting up the key combinatorial idea for the argument: Around this small neighborhood of the identity in $\hat{\mathbb{Z}}^2$, $\mathrm{SL}_2(\mathbb{Z})$ acts in a nearly linear fashion, resembling its canonical action on $\mathbb{R}^2$. The ultimate punch line of the proof is that the only "almost" $\mathrm{SL}_2(\mathbb{Z})$-invariant probability measures on $\mathbb{R}^2$ are those that largely concentrate with point mass at $\vec{0}$.

Then we can also think of $\mathrm{SL}_2(\mathbb{Z})$ as acting on the base space $\hat{\mathbb{Z}}^2$. By 2.6 (iv) we get at a.e. χ

$$\|((M^{-1}, \vec{0}) \cdot v)_{M^t \chi}\| = \|v_\chi\|.$$

Now forget everything else in the space $\mathcal{H}$ and just consider the direct integral presentation of v. From this we can define a measure

$$m(A) = \int_A \|v_\chi\| d\nu(\chi).$$

The almost invariance of v implies almost invariance of m.

m concentrates on the block

$$B = \{\chi_{(s,t)} : |s|, |t| < \delta\}.$$

B in turn can be broken up into four pieces, B_1, B_2, B_3, and B_4, depending on their quadrant in the plane. At each $i \leq 4$ let m_i be the restriction of m to the set B_i.

Consider the first of these.

$$B_1 = \{\chi_{(s,t)} : 0 \leq s, t < \delta\}.$$

Note then that

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \cdot B_1 \cap \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdot B_1 = \vec{0},$$

and hence, since these two matrices both map the first quadrant of $\mathbb{R}^2$ into itself, almost invariance of m implies that $m|_{B_1}$ can only have a very small amount of mass away from the point $\vec{0}$. With different uses of the matrices in our finite set, we obtain the same result for $m|_{B_i}$ all $i \leq 4$.

Thus m concentrates largely on $\vec{0}$. This in turn implies that $v(\{\vec{0}\}) \neq 0$, and we obtain a $\mathbb{Z}^2$-invariant vector in $\mathcal{H}$ with the measurable assignment

$$\chi_{(0,0)} \mapsto v_{\chi(0,0)}$$

and

$$\chi \mapsto 0$$

for

$$\chi \neq \chi_{(0,0)}.$$

□

2.4 Gaboriau-Popa

We are now in a position to prove Gaboriau-Popa, and I am going to present it in a way which disguises its roots in another area.

There is an essential point of view which motivates and inspires the construction; and while this point of view is thought of as absolutely trivial and not worthy of comment or even conscious recognition in the area of *operator algebras*, it is initially remarkable and surprising to mathematicians from vastly different fields.

From this perspective, every space is thought of as being an abelian object in a much more general kind of category. For instance, if (X, μ) is a measure space, then we associate it to $L^\infty(X, \mu)$, viewed as a group with the operation of pointwise addition.

And as a kind of converse, every abelian group is thought of as a space in waiting – given a countable group Δ we can form $\hat{\Delta}$, the compact group of all homomorphisms from Δ to $\{\xi \in \mathbb{C} : |\xi| = 1\}$ with group operation

$$\chi_1 \cdot \chi_2(\delta) = \chi_1(\delta)\chi_2(\delta).$$

We can in turn equip that dual group $\hat{\Delta}$ with its Haar measure ν, and view Δ as sitting inside $L^\infty(\hat{\Delta}, \nu)$ identifying each $\delta \in \Delta$ with the function

$$\chi \mapsto \chi(\delta).$$

It is that point of view which makes it possible to link the *rigidity*, the relative property (T), of $\mathbb{Z}^2$ inside $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ to the rigidity of certain kinds of measure preserving actions of $\mathrm{SL}_2(\mathbb{Z})$, and then for certiain copies of the free group lying inside.

Theorem 2.8 *(Gaboriau-Popa; [5]) For $n \geq 2$, $\mathbb{F}_n$ has continuum many orbit inequivalent measure preserving, free, ergodic actions on standard Borel probability spaces.*

Proof I will sketch an argument for $n = 3$ based around Asger Törnquist's thesis – see [27].

First of all, we need to represent $\mathbb{F}_2 = \langle a, b \rangle$ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ still has relative property (T) over $\mathbb{Z}^2$.

One way to do this is as follows. We take

$$a = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},$$

$$b = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

The argument from the last section still gives relative property (T), since we only used these elements and their inverses in $\mathrm{SL}_2(\mathbb{Z})$, but we need to see that they freely generate the subgroup $\angle a, b\rangle$. For this it suffices to see that reduced words of the form

$$W = a^{k_{2n-1}} b^{k_{2n-2}} .. b^{k_2} a^{k_1}$$

or

$$W = b^{k_{2n}} a^{k_{2n-1}} b^{k_{2n-2}} .. b^{k_2} a^{k_1}$$

are never equal to the identity. This can be shown by induction on n. Writing

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

we obtain that in the first case $|w_{12}|$ is larger than all the other $|w_{ij}|$'s, while in the second case $|w_{22}|$ is the dominant entry.

From now on I will identify $\mathbb{F}_2$ with the collection of matrices we generate in $\mathrm{SL}_2(\mathbb{Z})$ from

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

Now let $X = \hat{\mathbb{Z}^2} \cong \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ and equip it with the canonical probability measure μ from before. Let $L(\hat{\mathbb{Z}^2}, S^1, \mu)$ be the collection of measurable functions from $\hat{\mathbb{Z}^2} = X$ to complex numbers of norm 1. This becomes a Polish group with the metric

$$d(\phi, \psi) = \int |\phi(\chi) - \psi(\chi)| d\mu(\chi),$$

provided we identify functions agreeing a.e. We have a natural embedding

$$\mathbb{Z}^2 < L(\hat{\mathbb{Z}^2}, S^1, \mu),$$

where we think of $\vec{a} \in \mathbb{Z}^2$ as being identified with the function

$$\vec{a} : \chi \mapsto \chi(\vec{a}).$$

Let $\mathbb{F}_2 < \mathrm{SL}_2(\mathbb{Z})$ act on $\hat{\mathbb{Z}^2}$ in the manner suggested previously,

$$M : \chi_{(s,t)} \mapsto \chi_{M(s,t)},$$

and recall we have a certain intertwining of the action of $\mathbb{F}_2$ and $\mathbb{Z}^2$:

$$(M \cdot \vec{a})(\chi_{(s,t)}) = \vec{a}(\chi_{M^t \cdot (s,t)}).$$

Some properties of the action of $\mathbb{F}_2$ on $\hat{\mathbb{Z}^2}$:

(a) *It is free a.e.* This follows because the fixed points for some $M \in \mathbb{F}_2$ would correspond to eigenvectors for M, and only the identity element in $\mathbb{F}_2$ has a two dimensional collection of eigenvalues.
(b) *The action is ergodic.* This is a little bit more subtle. The main point here is that the functions arising from $\mathbb{Z}^2 \subset L(\hat{\mathbb{Z}^2}, S^1, \mu)$,

$$\vec{a} : \chi \mapsto \chi(\vec{a}),$$

form an orthonormal basis of the Hilbert space of $L^2(\hat{\mathbb{Z}^2}, \mu)$. It is a simple calculation to see that they orthonormal, and then to see they are a basis we appeal to Stone-Weierstrass: They separate points and thus the algebra they generate will be dense in the collection of continuous functions on $\hat{\mathbb{Z}^2}$, and hence dense in $L^2(\hat{\mathbb{Z}^2}, \mu)$.

Now given any non-constant $\phi \in L^2(\hat{\mathbb{Z}^2}, \mu)$ we want to show it is not invariant under the action of $\mathbb{F}_2$. This will suffice to prove ergodicity.

We write it in terms of the orthonormal basis:

$$\phi = \sum_{\mathbb{Z}^2} c_{\vec{a}} \vec{a},$$

for some choice of $c_{\vec{a}}$'s in $\mathbb{C}$. Invariance would imply invariance of the sets $A_c = \{\vec{a} \in \mathbb{Z}^2 : c_{\vec{a}} = c\}$ under the action of $\mathbb{F}_2$.

Now note that each A_c must be finite, since the Hilbert space norm of $\sum_{A_c} c\vec{a}$ equals $\sqrt{(|A_c|c^2)}$. The only finite subset of $\mathbb{Z}^2$ fixed under the action of $\mathbb{F}_2$ is $\{\vec{0}\}$, and thus we have $f = c\vec{0}$ for some $c \in \mathbb{C}$.

Now consider the Polish space $M_\infty(X, \mu)$, consisting of all measure preserving bijections from X to itself, with subbasic open sets consisting of $\{\phi \in M_\infty(X, \mu) : \mu(A\Delta\phi[B])\} < \epsilon$, for A, B measurable, $\epsilon > 0$. We will need a result due to Törnquist. As a word of notation, for groups $G_1, G_2 < H$, we say that G_1 is *freely joined with* G_2, if for any sequence $a_1, a_2, ..., a_n$ of non-identity elements in G_1, and $b_1, ...b_n$ of non-identity elements in G_2, the elements

$$a_n b_n a_{n-1} .... a_1 b_1,$$

$$a_n b_n a_{n-1} .... a_1,$$

$$b_n a_{n-1} .... b_2 a_1,$$

$$b_n a_{n-1} .... b_2 a_1 b_1,$$

are all unequal to the identity in H. In other words, the subgroup generated by G_1 and G_2 is naturally isomorphic to $G_1 * G_2$.

Theorem 2.9 *(Törnquist, [27]) For any countable $\Delta < M_\infty(X, \mu)$, the collection of $\varphi \in M_\infty(X, \mu)$ such that $\{\varphi^\ell : \ell \in \mathbb{Z}\}$ is freely joined with Δ is comeager.*

Applying that here, we in particular have that the collection of $\varphi \in M_\infty(X, \mu)$ such that

$$\langle a, b, \varphi \rangle$$

forms a free group in the indicated generators is comeager. Since the collection of ergodic transformations[2] is also comeager, we can obtain an ergodic $\varphi \in M_\infty(X, \mu)$ with $\langle a, b, \varphi \rangle$ generating a free group.

Now we will define a parameterized of measurable subsets $(B_t)_{0<t<1}$ of X such that

$$B_s \subset B_t, \ \ \mu(B_s) < \mu(B_t)$$

for $s < t$.

First fix a measurable partition $(A_n)_{n \in \mathbb{N}}$ of $X = \hat{\mathbb{Z}^2}$ with each $\mu(A_n) > 0$. Let $\{q_n : n \in \mathbb{N}\}$ be an enumeration of the rationals lying between 0 and 1. For each $t \in \mathbb{R}$ let

$$B_t = \bigcup \{A_n : q_n < t\}.$$

Note then that for $s < t$ we have $B_s \subset B_t$ and $B_t \setminus B_s$ non-null.

For each $t \in \mathbb{R}$ and $\chi \in \hat{\mathbb{Z}^2}$ we define $N_t(\chi)$ in one of two ways, depending on whether χ is in B_t: If $\chi \in B_t$ then $N_t(\chi)$ is the least $N > 0$ such that

$$\varphi^N(\chi) \in B_t;$$

[2] $\psi \in M_\infty(X, \mu)$ is *ergodic* if every measurable ψ-invariant set is either null or conull

if $\chi \in A_n$ for some n with $q_n \geq t$, then $N_t(\chi)$ is the least $N > 0$ such that

$$\varphi^N(\chi) \in A_n.$$

By the Poincare recurrence lemma, N_t is defined a.e.[3] We then define

$$\phi_t : X \to X$$

$$\chi \mapsto \varphi^{N_t(\chi)}(\chi).$$

ϕ_t is defined a.e. and the assignment

$$\mathbb{R} \to M_\infty(X, \mu),$$

$$t \mapsto \phi_t$$

is measurable. By considering the decomposition of the space as

$$X = B_t \dot{\cup} \dot{\bigcup_{q_n \geq t}} A_n$$

it can be shown that each ϕ_t is measure preserving and invertible.

We then let $a_t : \mathbb{F}_3 \times \mathbb{T}^2 \to \mathbb{T}^2$ be the action defined by

$$a_t(a, \chi) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdot \chi,$$

$$a_t(b, \chi) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \cdot \chi,$$

$$a_t(c, \chi) = \phi_t(\chi).$$

Some properties of *these* morphisms

(a) At each t we still have that $\langle a, b, \varphi_t \rangle$ generates a copy of $\mathbb{F}_3$ in the indicated generators. The induced action of $\mathbb{F}_3$ is free a.e., measure preserving, and ergodic.

(b) If for $0 < t < 1$ we let E_t be the orbit equivalence relation induced by the action of $\langle a, b, \varphi_t \rangle$, then we obtain that for $0 < s_1 < s_2 < 1$, the equivalence relation E_{s_1} has *infinite index* in E_{s_2} – each E_{s_2} equivalence class contains infinitely many E_{s_1} equivalence classes.

To see (b), Let C be the set of y for which $\{\phi^\ell_{s_1}(y) : \ell \in \mathbb{Z}\}$ is strictly included in $\{\phi^\ell_{s_2}(y) : \ell \in \mathbb{Z}\}$. It is non-null by construction. The saturation of C under a and b will be co-null by the ergodicity of their action. For a.e. x we can find an infinite set $\{y_n : n \in \mathbb{N}\} \subset C$ included in the orbit under $\langle a, b \rangle$ of x. Each of these y_n's in C will have its orbit under ϕ_{s_1} strictly included in its orbit under ϕ_{s_2}. Freeness of the action of $\langle a, b, \varphi \rangle$ then implies that for $y \in [y_n]_{\phi_{s_2}} \setminus [y_n]_{\phi_{s_1}}$ we have y not an element of $[y_m]_{\langle a, b, \phi_{s_1} \rangle}$ for any $m \neq n$.

Claim: Each E_s is orbit equivalent to only countably many other E_t's.

Proof of Claim: If not, let $W \subset [0, 1]$ be uncountable such that for some fixed $t \in (0, 1)$ we have at each $s \in W$ that E_s orbit equivalent to E_t via

$$\theta_s : X \to X,$$

[3] The Poincare recurrence lemma states that if (Y, m) is a standard Borel probability space, $T : Y \to Y$ measure preserving,, and $B \subset Y$ non-null, then almost every point in B meets it again under the forward orbits of T. This is easily seen in the case T is invertible, since the failure of the lemma would enable us to find a positive measure set C whose forward orbits never meet C; that in turn gives $\{T^n[C] : n \geq 0\}$ as a sequence of disjoint sets with equal measure – contradicting the finiteness of the measure.

a measurable bijection.

At each $s \in W$ define

$$\alpha_s : \mathbb{F}_3 \times X \to \mathbb{F}_3$$

by the requirement

$$\alpha(\gamma, \chi) \cdot \theta_s(\chi) = \theta_s(\gamma \cdot \chi).$$

By the separability of the space of measurable functions from $\mathbb{F}_3 \times X$ to $\mathbb{F}_3$ and the separability of $L(\hat{\mathbb{Z}^2}, S^1, \mu)$ we obtain $s_1 \neq s_2$ in W such that:

(a) $\alpha_{s_1}(\gamma, \chi) = \alpha_{s_2}(\gamma, \chi)$ for a large measure set of $\chi \in X$ and all $\gamma \in \mathbb{F}_2 = \langle a, b \rangle$ corresponding to a matrix of the form

$$\begin{bmatrix} 1 & \pm 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 3 & 1 \end{bmatrix};$$

(b) for $\vec{a} \in \mathbb{Z}^2$ of the form

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we have $\vec{a}(\theta_{s_1}(\chi)) \sim \vec{a}(\theta_{s_2}(\chi))$ for a large measure set of $\chi \in X$.[4]

Let $\theta = \theta_{s_2}^{-1} \theta_{s_1}$. This witnesses E_{s_1} orbit equivalent to E_{s_2}.

Now let $\mathcal{H}$ be the collection of all functions ψ from X to $\ell^2(X)$ with the following two properties:

1. for each $\chi \in X$
$$\psi(\chi) \in \ell^2([\theta(\chi)]_{E_{s_2}});$$
2. ψ is *measurable* in the sense that for any $w \in \langle a, b, \phi_{s_2} \rangle$ and open $O \subset \mathbb{C}$, the set
$$\{\chi \in X : \langle \psi(\chi), w \cdot \chi \rangle \in O\}$$
is measurable in X.

Provided we identify functions agreeing a.e, this becomes a Hilbert space with inner product

$$\langle \psi, \phi \rangle = \int_X \langle \psi(\chi), \phi(\chi) \rangle d\mu(\chi)$$

Let $\mathbb{Z}^2$ act on $\mathcal{H}$ by

$$(\vec{a} \cdot \psi)(\chi)(\xi) = \vec{a}(\chi) \vec{a}(\xi)^{-1} \psi(\chi)(\xi).$$

Let $\mathbb{F}_2 < \mathrm{SL}_2(\mathbb{Z})$ act on $\mathcal{H}$ by

$$(M \cdot \psi)(\chi)(\xi) = \psi(M^t \cdot \chi)(M^t \cdot \xi).$$

It follows from the intertwining property

$$(M \cdot \vec{a})(\chi_{(s,t)}) = \vec{a}(\chi_{M^t \cdot (s,t)})$$

that this indeed defines an action of $\mathbb{F}_2 \ltimes \mathbb{Z}^2$.

Let $\phi_\Delta \in \mathcal{H}$ be defined by

$$\phi_\Delta(\chi)(\theta(\chi)) = 1$$

but

$$\phi_\Delta(\chi)(\xi) = 0$$

[4]As a word on unpacking the notation, $\theta_s(\chi)$ will be a character, and then we use $\vec{a}(\theta_s(\chi))$ to denote $\theta_s(\chi)(\vec{a}) \in S^1$, the result of applying the character $\theta_s(\chi)$ to $\vec{a}$.

for $\xi \neq \theta(\chi)$.

Our assumptions on θ give that ϕ_Δ is almost invariant. By relative property (T), we obtain a non-trivial $\mathbb{Z}^2$-invariant vector.

Now we need an extra little trick. I claim we can choose the invariant vector so that it is close to ϕ_Δ.

One way to see this is to go back through the proof of theorem 2.7 and see that it indeed gives this sharper result. Another approach is to observe that whenever $H \lhd G$, with G having relative property (T) over H, and with G acting unitarily on some Hilbert space $\mathcal{W}$, then we can write $\mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_1$, where $\mathcal{W}_0$ is the collection of H-invariant vectors and $\mathcal{W}_1$ is its orthogonal complement. By relative property (T), no vector in $\mathcal{W}_1$ can be almost invariant, and thus every almost invariant vector is close to some vector in $\mathcal{W}_0$.

Granting this, we obtain $\mathbb{Z}^2$-invariant $\psi \in \mathcal{H}$ with $\psi \sim \phi_\Delta$. Thus on a positive measure set A we have for all $\chi \in A$

$$\exists! \zeta_\chi \in [\theta(\chi)]_{E_{s_2}}$$

such that

$$|\psi(\chi)(\zeta_\chi)| > \frac{3}{4}.$$

Then $\mathbb{Z}^2$ invariance gives for all $\chi \in A$,

$$(\vec{a} \cdot \chi)(\vec{a} \cdot \zeta_\chi)^{-1} = 1,$$

$$\therefore \vec{a} \cdot \chi = \vec{a} \cdot \zeta_\chi.$$

But $\mathbb{Z}^2$ separates points in $X = \hat{\mathbb{Z}}^2$. Thus on A, $\chi = \zeta_\chi$. Since $\zeta_\chi \in \theta[[\chi]_{E_{s_1}}]$ and θ effects an orbit equivalence,

$$E_{s_1}|_A = E_{s_2}|_A.$$

Since every E_{s_2} class contains infinitely many E_{s_1} classes and any non-null measurable set, such as A, is a selector for the ergodic equivalence relation, E_{s_1}, this gives a contradiction.

(□Claim)

□

2.5 But are we really using the spectral theorem?

No, not really.

Here is another way to think about the entire argument.

Definition Let Γ be a countable group acting on metric space (X, d). We say that the action is *expansive* if there is some finite $F \subset \Gamma$, and $\epsilon > 0$, and a finite collection of subsets

$$A_1, A_2, ...A_n \subset X \times X$$

such that:

(a) $\bigcup_{i \leq n} A_i \supset \{(x, y) \in X^2 : d(x, y) < \epsilon\}$;

(b) for each A_i there is a finite subset $F_i \subset F$ such that

$$\forall j \neq i (F_i \cdot A_i \cap A_j) \subset \Delta(X) =_{df} \{(x, x) : x \in X\}$$

and

$$\bigcap_{\gamma \in F_i} \gamma \cdot A_i \subset \Delta(X).$$

It is easily verified that the action of

$$\langle \begin{bmatrix} 1 & \pm 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 3 & 1 \end{bmatrix} \rangle \cong \mathbb{F}_2$$

on $\mathbb{R}^2/\mathbb{Z}^2$ is expansive in any choice of a compatible metric. This was basically the proof we went through in the course of 2.7 – we take $n = 4$ and let

$$A_1 = \{((s,t),(s',t') : 0 \leq s - s' \leq \frac{1}{100}, 0 \leq t - t' \leq \frac{1}{100}\},$$

$$A_2 = \{((s,t),(s',t') : -\frac{1}{100} \leq s - s' \leq 0, 0 \leq t - t' \leq \frac{1}{100}\},$$

$$A_3 = \{((s,t),(s',t') : 0 \leq s - s' \leq \frac{1}{100}, -\frac{1}{100} \leq t - t' \leq 0\},$$

$$A_4 = \{((s,t),(s',t') : -\frac{1}{100} \leq s - s' \leq 0, -\frac{1}{100} \leq t - t' \leq 0\}$$

and the argument devolves in to an exercise in linear algebra for F the set of matrices of the form

$$\begin{bmatrix} 1 & \pm 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 3 & 1 \end{bmatrix}.$$

Abstracting from the proof given above for Gaboriau-Popa we obtain theorems such as:

Theorem 2.10 *Let (X,d) be a complete, separable metric space equipped with an atomless Borel probability measure μ. Suppose Γ acts ergodically by measure preserving transformations on (X,μ) and the action on (X,d) is expansive. Let $(E_t)_{0<t<1}$ be a collection of distinct countable Borel equivalence relations on X with:*
(a) on any non-null measurable $A \subset X$ we have for all $s \neq t$ and a.e. $x \in A$

$$\{y \in A : yE_s x\} \neq \{y \in A : yE_t x\};$$

(b) each $E_s \supset E_\Gamma$.
Then each E_s is orbit equivalent to only countably many E_t's.

I mention this not as an exercise in proof mining, but rather because it might be interesting to explore dynamical properties such as expansiveness. Conceivably there could be other applications to the theory of countable Borel equivalence relations.

For instance, just for example, I do not know which countable groups admit measure preserving expansive actions.

2.6 Countable Borel equivalence relations up to Borel reducibility

For a long while we only had finitely many countable Borel equivalence relations which were known to be distinct in the $\leq_B$ ordering. This was finally settled by Adams and Kechris:

Theorem 2.11 *(Adams-Kechris, [1]) There are continuum many $\leq_B$-incomparable countable Borel equivalence relations.*

Their argument ultimately relied on super rigidity results for certain classes of algebraic groups, as developed in [29]. For the free groups there is no hint that such a rigidity theory should exist, and by implication there is not even the slightest suggestion that there is a parallel theory for treeable equivalence relations.

Definition A countable Borel equivalence relation E on a standard Borel space X is *treeable* if there is a symmetric, irreflexive Borel relation $R \subset X \times X$ such that

(a) R is acyclic. (If $x_0, x_1, ...x_n$ is a path with each $(x_i, x_{i+1}) \in R$ with no $x_i = x_j$ for $1 \leq i < j < n$, then $x_1 \neq x_n$).

(b) The R-connected components form the E-classes.

Definition For E, F Borel equivalence relations on Polish X, Y, E is *strongly Borel reducible to* F, written

$$E \leq_B^s F,$$

if there is a Borel function

$$\theta : X \to Y$$

such that

$$\{\theta(z) : zEx\} = [\theta(x)]_F;$$

in other words, we can witness Borel reducibility with a function whose image is F-invariant.

With minor adjustment, we can obtain from Gaboriau-Popa style arguments that:

Theorem 2.12 *There are continuum many treeable, countable Borel equivalence relations up to* $\leq_B^s$.

Going back to the proof we had before, it suffices to show that each E_t has only countably many E_s's with

$$E_s \leq_B^s E_t.$$

For a contradiction, suppose otherwise. The argument for Gaboriau-Popa allows us then to assume that there is an uncountable collection of s's with

$$\theta_s : X \to X$$

witnessing $E_s \leq_B^s E_t$ and the images are essentially disjoint – that is, for $s \neq s'$ we have conull $A \subset X$ with

$$\theta_s[A] \cap \theta_{s'}[A] = 0.$$

For convenience, let's assume outright that the images are always disjoint.

Then we can find $s_1 < s_2$ and

$$\theta_1, \theta_2 : X \to X$$

such that on a large measure set of $x \in X$ and for a large finite collection of $M \in \mathrm{SL}_2(\mathbb{Z})$:

(a) There will be a common $\gamma \in \langle a, b, \varphi \rangle$ such that

$$\theta_1(\gamma \cdot x) = M \cdot \theta_1(x),$$

$$\theta_2(\gamma \cdot x) = M \cdot \theta_2(x).$$

For future reference, if $\gamma \in \langle a, b, \varphi \rangle$ has the property that

$$\theta_i(\gamma \cdot x) = M \cdot \theta_i(x),$$

then we will denote by it $\gamma_{M,i,x}$.

We admittedly haven't set this up so the choice of $\gamma_{M,i,x}$ will be unique, but standard selection theorems enable us to choose such a group element for each $\theta_i(x)$ and M, and let us go forward assuming this has been put in to place and we obtain the indicated agreement on a large set.

(b) For $\vec{a} \in \mathbb{Z}^2$ of the form

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we obtain

$$\vec{a} \cdot (\theta_1(x)) \sim \vec{a}(\theta_2(x)).$$

Now we take our Hilbert space to be the collection of functions

$$\phi : X \to \ell^2(X) \times \ell^2(X)$$

with $\phi(x) \in \ell^2([\theta_1(x)]_{E_{s_1}}) \times \ell^2([\theta_2(x)]_{E_{s_2}})$ all $x \in X$. We let $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}$ act with

$$(M, \vec{0}) \cdot \phi(\gamma_{(M^t)^{-1},1,x} \cdot x)((M^t)^{-1} \cdot y_1)((M^t)^{-1} \cdot y_2) = \phi(x)(y_1)(y_2)$$

and

$$\vec{a} \cdot \phi(x)(y_1)(y_2) = \vec{a}(y_1)\vec{a}(y_2)^{-1}\phi(x)(y_1)(y_2).$$

Since $\theta_1(\gamma \cdot x) = M \cdot \theta_1(x) = \theta_2(\gamma \cdot x) = M \cdot \theta_1(x)$ for a large finite collection of M and for a large measure set of $x \in X$, we can obtain our almost invariant vector with

$$\phi^+_\Delta(x)(y_1)(y_2) = 1$$

if $y_i = \theta_i(x)$, and

$$\phi^+_\Delta(x)(y_1)(y_2) = 0$$

otherwise. Then as before we obtain a $\mathbb{Z}^2$-invariant vector which is close to ϕ^+_Δ. As before, the invariance will imply agreement on ranges, with the same contradiction as previously.

Question Does every countable non-amenable group give rise to continuum many free, measure preserving, ergodic actions up to strong Borel reducibility?

2.7 Further results on orbit equivalence for non-amenable groups

From about 2004 until about 2007 there were a sequence of partial results towards trying to prove that all non-amenable groups have continuum many orbit inequivalent measure preserving, free, ergodic actions on standard Borel probability spaces.

Theorem 2.13 *(Popa, [23]) If Γ is a countable group having a relative property (T) over an infinite normal subgroup, then it has continuum many orbit inequivalent measure preserving, free, ergodic actions on standard Borel probability spaces.*

Theorem 2.14 *(Monod-Shalom, [21]) If Γ is a countable free product of non-amenable hyperbolic*[5] *groups, then it has has continuum many orbit inequivalent measure preserving, free, ergodic actions on standard Borel probability spaces.*

Theorem 2.15 *(Ioana, [12]) If Γ is a countable product of a non-amenable group and an infinite amenable group, then it has continuum many orbit inequivalent measure preserving, free, ergodic actions on standard Borel probability spaces.*

Theorem 2.16 *(Ioana, [13]) If Γ is a countable group containing $\mathbb{F}_2$ as a subgroup, then it has continuum many orbit inequivalent measure preserving, free, ergodic actions on standard Borel probability spaces.*

The most general result possible was finally obtained by Inessa Epstein in 2007. Before presenting a sketch of her proof, we need to spend some time discussing the notion of cost.

[5] A group is said to be *hyperbolic* or *word hyperbolic* if there is a finite generating set and the induced Cayley graph with the induced path metric is hyperbolic as a space – that is to say there is some $N \in \mathbb{N}$ such that in any geodesic triangle, each side is contained within the ball of radius N around the other two.

3 Lecture III

3.1 Introduction to the theory of cost

For technical reasons we will want to spend some time working with measure spaces which are finite without necessarily having total measure 1. Many of the unattributed results in this section can be found in [19].

Definition Let E be a countable Borel, measure preserving equivalence relations on a standard finite measure space, (X, μ). A *graphing* of E is a *graphing countable collection* Φ *such that:*

(a) each $\phi \in \Phi$ *is a bijection of the form*

$$\phi : A_\phi \to B_\phi$$

for A_ϕ, B_ϕ *measurable sets, with*

$$xE\phi(x)$$

all $x \in A_\phi$*;*

(b) for any $(x, y) \in E$ *there is a sequence* $\phi_0, \phi_1, ...\phi_n \in \Phi \cup \Phi^{-1}$ *with* $y = \phi_n \circ \phi_{n-1} \circ ...\phi_0(x)$*. (Here* Φ^{-1} *is the collection of morphisms* $\phi^{-1} : B_\phi \to A_\phi$*.)*

Φ *is a treeing of* E *if for* μ *almost every* $x \in X$ *we have that for all* yEx *there is a unique choice of* $\phi_0, \phi_1, ...\phi_n \in \Phi \cup \Phi^{-1}$ *with* $y = \phi_n \circ \phi_{n-1} \circ ...\phi_0(x)$ *and*

$$\phi_i \circ \phi_{i-1} \circ ...\phi_0(x) \neq \phi_j \circ \phi_{j-1} \circ ...\phi_0(x)$$

all $i < j \leq n$.

We let $C_\mu(\Phi)$*, the cost of* Φ*, be*

$$\sum_{\phi \in \Phi} \mu(A_\phi).$$

We let $C_\mu(E)$*, the cost of* E*, be the infimum over* $C_\mu(\Phi)$ *as* Φ *ranges over graphings of* E*.*

Technically this will be the most useful definition of cost for our purposes. There is however another definition which is combinatorially more elegant.

Definition A Borel graph $R \subset X \times X$ is said to *graph* E if its connected components are the E-classes. We then define $C_\mu(R)$ to be

$$\int_X \frac{1}{2}\deg(x)d\mu(x),$$

where $\deg(x)$ denotes the number of edges incident to x. (If this $\deg(x)$ is infinite on a non-null set, then $C_\mu(R)$ is infinite).

The $\frac{1}{2}$ in this definition is to avoid counting a morphism twice when we use an R that graphs E to generate a graphing in the previous sense.

Lemma 3.1 *$C_\mu(E)$ equals the infimum of $C_\mu(R)$ as R ranges over Borel graphs which graph E.*

Lemma 3.2 *If the cost is attained by a specific graphing, $C_\mu(E) = C_\mu(\Phi)$ for Φ some graphing Φ of E, then Φ is a treeing.*

In a nutshell, here is why: If Φ fails to be a treeing, then on a non-null set there will be a specific reason for this failure. Namely, there will be some $A \subset X$ non-null and $\phi_0, \phi_1, ..., \phi_n$ such that every $x \in A$ has $\phi_n \circ \phi_{n-1} \circ ...\phi_0(x) = x$ as a witness to it not being a treeing. Then we can reduce A such that the sets of the form

$$A_i = \phi_i \circ \phi_{i-1} \circ ...\phi_0[A]$$

are disjoint. Then we can replace ϕ_0 by

$$\phi_0|_{A_{\phi_0} \setminus A}$$

and obtain a graphing with lower cost.

Lemma 3.3 *If E has every equivalence class of size N for some fixed $N \in \mathbb{N}$, then*

$$C_\mu(E) = \frac{N-1}{N}\mu(X).$$

Roughly speaking this holds because we can graph a set of size N with $N-1$ edges, but any choice of less than $N-1$ edges will leave it disconnected.

Lemma 3.4 *If E has finite classes, then we can choose a measurable selector*[6] *$A \subset X$ and for any such choice of A we have*

$$C_\mu(E) = \mu(X \setminus A).$$

This is a variation on the last lemma.

Lemma 3.5 *If all E classes are infinite, then*

$$C_\mu(E) \geq \mu(X).$$

In very rough terms, this lemma is true because an infinite connected graph must have infinitely many edges; however it does take some care and effort to turn that combinatorial fact into a proof of the measure theoretic statement at lemma 3.5.

Lemma 3.6 *If E is hyperfinite, then $C_\mu(E) \leq \mu(X)$.*

The idea of the proof of this lemma is based around an iterative construction: We build a sequence of partial morphisms

$$\phi_1 \subset \phi_2 \subset ...\phi_n \subset ...$$

such that at each n the set consisting of the n^{th} morphism, $\{\phi_n\}$, provides a treeing for E_n and each ϕ_n is injective. Given ϕ_n, then for each x we let

$$[y_{0,x}]_{E_n}, [y_{1,x}]_{E_n}, ..., [y_{m(n,x),x}]_{E_n}$$

enumerate the E_n-classes included in $[x]_{E_{n+1}}$. We can do this so that each $y_{i,x}$ is *not* in the domain of ϕ. We can also make the process sufficiently uniform that the sets of the form

$$B_k = \{x \in X : m(x, n) \geq k\}$$

are measurable and at each $i \leq k$ the function

$$B_k \to B_k$$

[6]That is to say, a set which meets each equivalence class in exactly one point

$$x \mapsto y_{i,x}$$

is measurable. We then let ϕ_{n+1} extend ϕ_n with

$$\phi_{n+1}(y_{i,x}) = y_{i+1,x}.$$

There is one more basic fact about cost which is somewhat more involved, and I mention it now only for later reference.

Theorem 3.7 *(The Cost Restriction Formula; Gaboriau, [4]) If E is a measure preserving, ergodic, countable equivalence relation on a standard finite measure space (X, μ), then for any measurable non-null $A \subset X$*

$$C_\mu(E|_A) = C_\mu(E) - \mu(X \setminus A).$$

The proof of this theorem resists easy summary, but I will say something about the rough idea. We start with a graphing Φ of E and we want to reorganize it into a new graphing of the same cost containing morphisms which spread A out across X.

In the simplest case, let us suppose $\mu(A) = \frac{1}{2}\mu(X)$. We will reorganize Φ so it has the form $\Phi = \{\phi\} \cup \Psi$ with $\phi : A \to X \setminus A$. Then we suitably post compose and or pre compose each element of Ψ by ϕ or its inverse to obtain Φ' of the same cost with $\Phi' \setminus \{\phi\}$ graphing $E|_A$. For instance if $\psi : A_\psi \to B_\psi$ is in Φ we can assume, after appropriately subdividing, that both of sets A_ψ and B_ψ are either disjoint from or included in A. If they are both disjoint from A, then we replace ψ by $\phi^{-1} \circ \psi \circ \phi$. If just A_ψ is disjoint from A, then we replace by $\psi \circ \phi$, and so on.

The hard part is to find ϕ. We build it by gluing together morphisms in Φ.

First of all, by ergodicity, there must be some $\psi \in \Phi$ and non-null $B \subset A_\psi \cap A$ such that

$$\psi[B] \subset X \setminus A.$$

Then we adjust Φ to obtain $\Phi_0 = \Phi \setminus \{\psi\} \cup \{\psi|_{A_\psi \setminus B}, \psi|_B\}$.

The next step will be more complicated, since there is a split in cases. A typical case might find some $C \subset A_{\phi_1}$ and ϕ_1, ϕ_2, ϕ_3 such that

$$C \subset A \setminus B,$$

$$\phi_1[C] \subset B,$$

$$\phi_2 \circ \phi_1[C] \subset \psi[A],$$

$$\phi_3 \circ \phi_2 \circ \phi_1[C] \subset X \setminus (A \cup \psi[A]).$$

Then we replace ϕ_1 in Φ_0 with

$$\phi_1|_{A_{\phi_1} \setminus C}$$

and $\psi|_B$ with

$$\psi|_B \cup (\phi_3 \circ \phi_2 \circ \phi_1)|_C.$$

And on from there with many slightly differing cases all to be accounted for with suitable variations of the procedure described here.

3.2 Gaboriau's breakthrough result

We will go through a sketch of a proof of Gaboriau's converse to 3.2, but before doing so I want to emphasize the dramatic originality of his argument.

The notion of cost was not first defined by Gaboriau, but it stood there as an isolated definition with no apparent applications. Specific calculations of cost appeared to be intractable – except in the relatively trivial cases of hyperfiniteness or finite classes.

In the neighborhood of this invariant many problems remained apparently impervious to any known method of attack. It had been an embarrassment for 30 years that no one knew whether the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ was orbit equivalent to the shift action of $\mathbb{F}_3$ on $2^{\mathbb{F}_3}$.

All this was settled by Gaboriau's theorem:

Theorem 3.8 *(Gaboriau, [4]) Let E be a countable, measure preserving, equivalence relation on a standard finite Borel measure space (X, μ). Let Φ be a treeing of E. Then*

$$C_\mu(E) = C_\mu(\Phi).$$

Proof I am going to sketch the proof in a special but representative case.

Let us first make the harmless assumption that $\mu(X) = 1$. I will also assume the action is ergodic; the more general argument would require us to first pass to the ergodic components of E in order to apply the cost restriction formula. Now let us also make the much more specific and drastic assumption that $E = E_{\mathbb{F}_2}$ is induced by a free measure preserving action of the free group, $\mathbb{F}_2 = \langle a, b \rangle$. The action provides a natural treeing of E, with $\{\phi_a, \phi_b\}$, where

$$\phi_a : x \mapsto a \cdot x$$

and

$$\phi_b : x \mapsto b \cdot x,$$

both defined on all of X. That graphing has cost 2. It suffices to see that no graphing can have lower cost.

Let Φ be some other graphing of E. We may assume without loss of generality that for each $\phi \in \Phi$ there is a specific $w_\phi \in \mathbb{F}_2$ such that on all $x \in A_\phi$ we have

$$\phi(x) = w_\phi \cdot x.$$

We may also assume that Φ is finite. The point is that if Φ provides a counterexample, with $C_\mu(\Phi) < 2$, then we can take some other graphing which is finite, arising from replacing infinitely many of the morphisms of Φ with ϕ_a and ϕ_b restricted to sufficiently small sets, which will still have cost less than 2.

We are going to expand X, obtaining a kind of covering space.

We let $\hat{X}$ consist of all triples (x, u, ϕ) where $\phi \in \Phi$ and u is a *right segment* of w_ϕ. We identify x with (x, e, ϕ) and $\phi(x)$ with (x, w_ϕ, ϕ). Thus between A_ϕ and B_ϕ we create a kind of bridge which passes through the copies of the points we should have traversed if we had instead used w_ϕ and the graphing provided by the action of $\mathbb{F}_2$.

Now let $\hat{\Phi}$ consist of the morphisms of the form

$$\{(x, u, \phi) : x \in A_\phi\} \to \{(x, cu, \phi) : x \in A_\phi\}$$

$$(x, u, \phi) \mapsto (x, cu, \phi),$$

for u and cu right segments of w_ϕ, $c \in \{a^{\pm 1}, b^{\pm 1}\}$. We define

$$p : \hat{X} \setminus X \to X$$

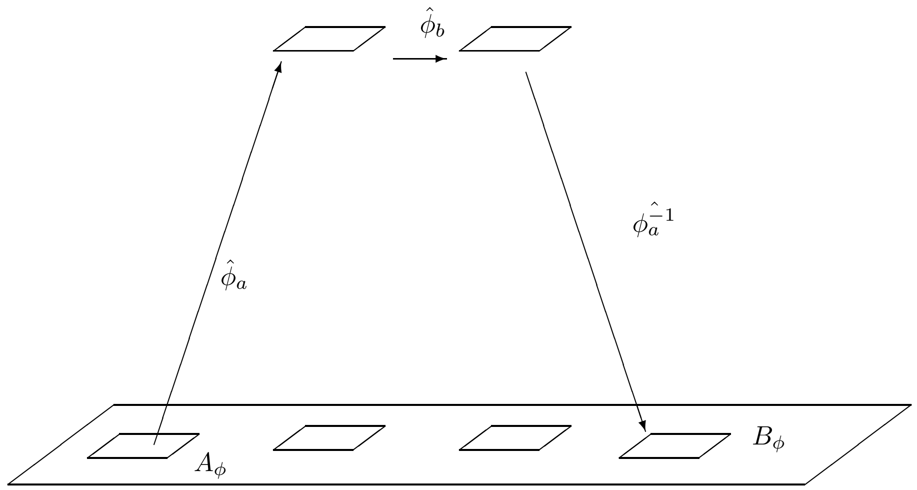

Figure 1: Here is the situation we would have in the event that $w_\phi = a^{-1}ba$.

by

$$(x, u, \phi) \mapsto u \cdot x.$$

We let $\hat{E}_a$ be the equivalence relation on $\hat{X}$ generated by p and the action of the generator $a \in \mathbb{F}_2$ – thus, $\hat{x}\hat{E}_a\hat{y}$ if for some $\ell \in \mathbb{Z}$

$$a^\ell \cdot p(\hat{x}) = p(\hat{y}).$$

Similarly $\hat{E}_b$ is generated from p and $b \in \mathbb{F}_2$.

We let $\hat{\mu}$ be the measure we obtain by extending μ to $\hat{X}$ with

$$\hat{\mu}(\{(x, u, \phi) : x \in A_\phi\}) = \mu(A_\phi).$$

Then by the cost restriction formula we have

$$C_{\hat{\mu}}(\hat{\Phi}) = C_\mu(\Phi) + \hat{\mu}(\hat{X} \setminus X).$$

Thus, it suffices to show $C_{\hat{\mu}}(\hat{\Phi}) \geq 2 + \hat{\mu}(\hat{X} \setminus X)$.

Since $\hat{E}_a$ and $\hat{E}_b$ have infinite classes, they both have cost at least equal to

$$1 + \hat{\mu}(\hat{X} \setminus X) = \hat{\mu}(\hat{X}).$$

Thus it will suffice for us to prove that

$$C_{\hat{\mu}}(\hat{\Phi}) \geq C_{\hat{\mu}}(\hat{E}_a) + C_{\hat{\mu}}(\hat{E}_b) - \hat{\mu}(\hat{X} \setminus X).$$

We let E_p be the equivalence relation generated by p. Note then that it has cost equal to $\hat{\mu}(\hat{X} \setminus X)$ by 3.4. We let $\hat{\Phi}_a$ be the morphisms in $\hat{\Phi}$ of the form

$$(x, u, \phi) \mapsto (x, cu, \phi),$$

for $c = a^{\pm 1}$, and $\hat{\Phi}_b$ be the morphisms in $\hat{\Phi}$ of the form

$$(x, u, \phi) \mapsto (x, cu, \phi),$$

for $c = b^{\pm 1}$.

Claim: There exist Ψ_a, Ψ_b such that:

(a) Ψ_a, Ψ_b are disjoint;

(b) $\Psi_a \cup \Psi_b$ is a treeing of a subequivalence relation of E_p.

(c) If we let $F_a = E_{(\hat{\Phi}_a \cup \Psi_a)}$ be the equivalence relation generated by the collection or morphisms $\hat{\Phi}_a \cup \Psi_a$, and similarly set $F_b = E_{(\hat{\Phi}_b \cup \Psi_b)}$, then

$$F_a \cap E_p = F_b \cap E_p.$$

Proof of Claim: We build a sequence of collections of morphisms

$$\Psi_a^0 \subset \Psi_a^1 \subset \Psi_a^2 \subset ...$$

$$\Psi_b^0 \subset \Psi_b^1 \subset \Psi_b^2 \subset ...,$$

commencing with $\Psi_a^0 = \Psi_b^0 = \emptyset$, both empty, and maintaining at each stage:

(i) $\Psi_a^n \cap \Psi_0^b = \emptyset$;

(ii) every morphism in $\Psi_a^{n+1} \setminus \Psi_a^n$ connects points which are equivalent under the equivalence relation

$$E_{(\hat{\Phi}_b \cup \Psi_b^n)} \cap E_p;$$

and moreover

$$E_{(\hat{\Phi}_b \cup \Psi_b^n)} \cap E_p \subset E_{(\hat{\Phi}_a \cup \Psi_a^{n+1})};$$

(iii) every morphism in $\Psi_b^{n+1} \setminus \Psi_b^n$ connects points which are equivalent under the equivalence relation

$$E_{(\hat{\Phi}_a \cup \Psi_a^{n+1})} \cap E_p;$$

moreover

$$E_{(\hat{\Phi}_a \cup \Psi_a^{n+1})} \cap E_p \subset E_{(\hat{\Phi}_b \cup \Psi_b^{n+1})};$$

(iv) $\Psi_a^n \cup \Psi_b^n$ is a treeing of a subequivalence relation of E_p;

(iv) $\Psi_a^{n+1} \cup \Psi_b^n$ is a treeing of a subequivalence relation of E_p.

Clearly if we maintain this, then taking $\Psi_a = \bigcup_{n \in \mathbb{N}} \Psi_a^n$ and $\Psi_b = \bigcup_{n \in \mathbb{N}} \Psi_b^n$ completes the proof of the claim.

We first start working on Ψ_a^1: The equivalence relation

$$E_{(\hat{\Phi}_a \cup \Psi_a^0)} \cap E_p = E_{\hat{\Phi}_a} \cap E_p$$

obviously has finite classes. If no two disjoint equivalence classes contain elements which are

$$E_{(\hat{\Phi}_b \cup \Psi_b^0)} \cap E_p = E_{\hat{\Phi}_b} \cap E_p$$

equivalent, then there is no further work to do and we let $\Psi_a^1 = \Psi_a^0 = \emptyset$.

In the case that there are distinct classes, say

$$[x_0]_{E_{\hat{\Phi}_a} \cap E_p},$$

$$[x_1]_{E_{\hat{\Phi}_a} \cap E_p},$$

$$...$$

$$[x_m]_{E_{\hat{\Phi}_a} \cap E_p},$$

which are linked in $E_{\hat{\Phi}_b} \cap E_p$, we choose exactly one point

$$y_0 \in [x_0]_{E_{(\hat{\Phi}_a \cap E_p)}},$$

$$y_1 \in [x_1]_{E_{(\hat{\Phi}_a \cap E_p)}},$$

$$...$$

$$y_m \in [x_m]_{E_{\hat{\Phi}_a} \cap E_p},$$

which are $E_{\hat{\Phi}_b} \cap E_p$-equivalent and require that some morphisms link them in Ψ^1_a. We can do this sparingly and in a way which does not introduce a cyclic amongst the various equivalence classes, for instance setting $\phi(y_i) = y_{i+1}$, and only being defined on $y_0, ..., y_{m-1}$ in this E_p-equivalence class.

(Of course there are various selection problems, and we need to argue that all these tasks can be completed in a Borel manner, giving only countably many new morphisms in Ψ^1_a. These steps use nothing beyond the classical theory of Borel sets and measurable functions, and I omit them. The key point here is that E_p has finite classes.)

Now for the other side. We consider the equivalence relation,

$$E_{(\hat{\Phi}_b \cup \Psi^0_b)} \cap E_p = E_{\hat{\Phi}_b} \cap E_p,$$

which again has finite classes. Again we consider a sequence of classes

$$[x_0]_{E_{\hat{\Phi}_b} \cap E_p},$$

$$[x_1]_{E_{\hat{\Phi}_b} \cap E_p},$$

$$...$$

$$[x_m]_{E_{\hat{\Phi}_b} \cap E_p},$$

which contain elements which are linked in

$$E_{(\hat{\Phi}_a \cup \Psi^1_a)} \cap E_p;$$

we choose elements

$$y_0 \in [x_0]_{E_{(\hat{\Phi}_b \cap E_p)}},$$

$$y_1 \in [x_1]_{E_{(\hat{\Phi}_b \cap E_p)}},$$

$$...$$

$$y_m \in [x_m]_{E_{\hat{\Phi}_b} \cap E_p},$$

and require one of the morphisms in Ψ^1_b to provide links without introducing cycles.

Here we do need to check that $\Psi^1_a \cup \Psi^1_b$ is still a treeing. This amounts to checking that there is no morphism in Ψ^1_a linking any of the

$$[x_i]_{E_{\hat{\Phi}_b} \cap E_p}$$

to

$$[x_j]_{E_{\hat{\Phi}_b} \cap E_p}$$

for $i \neq j$ in the above situation. But if there was such a link it would have to have been in $E_{\hat{\Phi}_b} = E_{\hat{\Phi}_b \cup \Phi_n^0}$ by the assumption that every morphism in $\Psi_a^1 \setminus \Psi_a^0$ connects points which are equivalent under the equivalence relation

$$E_{(\hat{\Phi}_b \cup \Psi_b^0)} \cap E_p.$$

Now we go back to the Ψ_a^1 side. We look for disjoint equivalence classes which are linked in

$$E_{(\hat{\Phi}_b \cup \Psi_b^1)} \cap E_p.$$

We add links as needed. Again we do not introduce cycles, since the the equivalence classes which demand work cannot be connected under Ψ_b^1 since every morphism in $\Psi_b^1 \setminus \Psi_b^0$ already appeared in $E_{(\hat{\Phi}_a \cup \Psi_a^1)} \cap E_p$.

And so on. (□Claim)

Claim: $\Psi_a \cap \hat{\Phi}_a$ is a graphing of $\hat{E}_a$.

Proof of Claim: Suppose $z\hat{E}_a z'$. Then we can certainly link them in $\hat{\Phi}$, and in particular we can find a chain

$$z_0 = z, z_1, ...z_n = z'$$

where each $(z_i, z_{i+1}) \in E_{(\hat{\Phi}_a \cup \Psi_a)}$ or in $E_{(\hat{\Phi}_b \cup \Psi_b)}$.

Let us assume n is minimal. That would involve strictly alternating between the two equivalence relations.

By considering $x_i = p(z_i)$ as a path through X, we note that either $n = 1$ or there is some pair $x_i = x_{i+1}$ – this goes back to the fact that $\mathbb{F}_2 = \langle a, b \rangle$ is acting freely. This gives then that (z_i, z_{i+1}) is in either $E_{(\hat{\Phi}_a \cup \Psi_a)} \cap E_p$ or in $E_{(\hat{\Phi}_b \cup \Psi_b)} \cap E_p$. But the assumptions on Ψ_a and Ψ_b imply that once it is one then it is in both, and we can shorten the length of the chain.

So now we have $n = 1$. But then we can again use $\hat{E}_a \cap \hat{E}_b = E_p$ to obtain $zE_{\Psi_a \cap \hat{\Phi}_a} z'$. (□Claim)

By exactly the same argument:

Claim: $\Psi_b \cap \hat{\Phi}_b$ is a graphing of $\hat{E}_b$.

$\Psi_a \cup \Psi_b$ is a treeing of a subequivalence relation of E_p. By the considerations raised at 3.3 and 3.4, we have that

$$C_{\hat{\mu}}(\Psi_a \cup \Psi_b) \leq \hat{\mu}(\hat{X} \setminus X).$$

We obtain

$$2 + 2\hat{\mu}(\hat{X} \setminus X) = 2\hat{\mu}(\hat{X}) \leq C_{\hat{\mu}}(\hat{E}_a) + C_{\hat{\mu}}(\hat{E}_b),$$

since $\hat{E}_a$ and $\hat{E}_b$ have infinite classes. Since $\hat{\Phi}_a \cup \Psi_a$ graphs $\hat{E}_a$ and $\hat{\Phi}_b \cup \Psi_b$ graphs $\hat{E}_b$ we obtain

$$C_{\hat{\mu}}(\hat{E}_a) + C_{\hat{\mu}}(\hat{E}_a) \leq C_{\hat{\mu}}(\hat{\Phi}_a \cup \Psi_a) + C_{\hat{\mu}}(\hat{\Phi}_b \cup \Psi_b) \leq C_{\hat{\mu}}(\hat{\Phi}) + C_{\hat{\mu}}(\Psi_a \cup \Psi_b)$$

which in turn is bounded by

$$C_{\hat{\mu}}(\hat{\Phi}) + \hat{\mu}(\hat{X} \setminus X),$$

exactly as required.

□

4 Lecture IV

4.1 More ramifications of cost

Before Gaboriau's result we knew almost nothing about cost. After, there was a flood of results.

From now on I will go back to the "real world", so to speak, of all our measures spaces having $\mu(X) = 1$.

Theorem 4.1 *(Kechris-Miller, [18]) Let E be a countable, measure preserving equivalence relation on a standard Borel probability space (X, μ) with*

$$C_\mu(E) > \alpha.$$

Then E contains a treeable sub-equivalence relation $F \subset E$ also with

$$C_\mu(F) > \alpha.$$

Theorem 4.2 *(Hjorth, [11]) Let E be a countable, measure preserving, equivalence relation on a standard Borel probability space (X, μ). Assume that E is treeable, ergodic, and $C_\mu(E) = n$, for some $n \in \{1, 2, 3, ..., \infty\}$.*

Then there is a free, measure preserving action of $\mathbb{F}_n$ on X such that on some conull set $A \subset X$

$$E|_A = E_{\mathbb{F}_2}|_A.$$

Corollary 4.3 *Let E be a measure preserving equivalence relation on a standard Borel probability space (X, μ) with $C_\mu(E) > 1$. Then there is a non-null $A \subset X$ and a free, Borel, measure preserving action of $\mathbb{F}_2$ on A with*

$$E_{\mathbb{F}_2} \subset E.$$

Proof Applying 4.1 we may assume that E is treeable. After passing to to the ergodic decomposition[7] it suffices to consider the case that E is ergodic. Applying the cost restriction formula we may find a set A with

$$C_{\mu|_A}(E|_A) = n\mu(A),$$

for some $n \in \{2, 3, ..., \infty\}$. We renormalize the measure with

$$\nu = \frac{\mu}{\mu(A)}$$

to obtain that (A, ν) is a standard Borel probability space.

Finishing with 4.2 we obtain a free action, measure preserving action of $\mathbb{F}_n$ on A with $E_{\mathbb{F}_n} = E|_A$. □

In fact, it is something of an overkill to apply 4.2 in obtaining this consequence of 4.1. The real point of 4.2 is not that treeable equivalence relations of high cost include orbit equivalence relations induced by free actions of non-abelian groups; it is rather that under certain circumstances they are exactly equal. As noted in [6], a careful analysis of the proof of 4.1 given at [18] will show that if $C_\mu(E) \geq 2$ then we obtain a free action of $\mathbb{F}_2$ with $E_{\mathbb{F}_2}$ included in the F they provide.

Notation For A a set, $\mathcal{P}(A)$ denotes the collection of all its subsets and $\mathcal{P}_{<\aleph_0}(A)$ denotes the collection of all its finite subsets.

[7]The ergodic decomposition states that any equivalence relation may be written as a *direct integral* or ergodic measure preserving transformations – in a way very parallel to the direct integral decomposition of a unitary representation in terms of irreducible representations. See for instance [28].

Definition Let R be a connected graph with underlying set V. An *end* of R is a function

$$\varphi : \mathcal{P}_{<\aleph_0}(V) \to \mathcal{P}(V)$$

such that:
(a) $\varphi(S)$ is always a connected component of the remainder, $(V \setminus S, R|_{(V\setminus S)^2})$.
(b) $\forall S_1, S_2 \in \mathcal{P}_{<\aleph_0}(V)(\varphi(S_1) \cap \varphi(S_2) \neq \emptyset)$.

Thus an end is a consistent way of choosing connected components whenever we remove a finite subset.

Definition An equivalence relation E on a standard Borel space X is said to *have infnitely many ends* if there is a Borel graphing $R \subset X \times X$ with every vertex having finite degree and such that R graphs E and on each equivalence class the induced graph has infinitely many ends.

Theorem 4.4 *(Gaboriau, [4]) If E is a measure preserving equivalence equivalence on a standard Borel probability space (X, μ) with infinitely many ends, then $C_\mu(E) > 1$.*

This result is a based on analysis by Ghys. I am not going to give a proof, but I will give a plausibility argument.

Using standard Borel techniques, we can assume that there is a Borel collection $\mathcal{C} \subset \mathcal{P}_{<\aleph_0}(X)$ such that:
(a) Each $C \in \mathcal{C}$ is a connected subgraph inside one of the equivalence classes.
(b) Any two elements of $\mathcal{C}$ are either in different equivalence classes, or they have distance at least 3 apart in the graphing.
(c) Every equivalence class includes some element of $\mathcal{C}$.
(d) And finally, $[x]_E \setminus C$ always has more than three component when $C \subset [x]_E$.

At each $C \in \mathcal{C}$, with say

$$C \subset [x]_E,$$

$$[x]_E \setminus C$$

having $n(C)$ many components, we choose $x_{C,0}, ..., x_{C,n(C)}$ in the distinct components outside C, each adjacent to C; it is possible to do this so that each assignment of the form

$$C \mapsto x_{C,i}$$

is Borel. We now restriction our attention the set B of all points of the form $x_{C,i}$ some $i \leq n(C)$.

Let us make the simplifying assumption that $n(C)$ is always 3. Then we let E_1 be the equivalence relation generated by the morphisms of the form

$$x_{C,0} \mapsto x_{C,1}$$

and E_2 the equivalence relation generated by morphisms of the form

$$x_{C,0} \mapsto x_{C,2}.$$

Finally we set xFy if they are E-equivalent and in the same connected component of $[x]_E \setminus \bigcup C$. Since the elements in $\mathcal{C}$ are always at least a certain set distance apart, every element in B is F-equivalent to some other point.

These equivalence relations will be freely joined, in the sense that there is no cycle of the form

$$x_0, x_1, .., x_n = x_0$$

with each (x_i, x_{i+1}) in one of the three equivalence relations, and then (x_{i+1}, x_{i+2}) in one of the other two.

E_1 and E_2 both have cost on $(B, \mu|_B)$ equal to

$$\frac{\mu(B)}{3}.$$

F has cost at least equal to

$$\frac{\mu(B)}{2}.$$

Thus with an appropriate appeal to a suitable generalization of theorem 3.8 to the cost of freely joined equivalence relations, which is indeed presently explicitly in [4], we obtain $\mu(E|_B) > \mu(B)$, and now we can derive $\mu(E) > 1$ by the cost restriction formula.

Question Is there a more direct proof of 4.4?

It remains very much open at the time of writing whether every countable, measure preserving, ergodic equivalence relation which is not hyperfinite contains an equivalence relation induced by a free action of $\mathbb{F}_2$. Several years ago Gaboriau and Lyons showed that for every non-amenable countable Γ there is *some* free measure preserving action which has this property. At the time they viewed it as simply a partial result towards this more general question, and did not bother to publish or even write it up as a preprint.

Theorem 4.5 *(Gaboriau-Lyons, [6]) Let Γ be a countable non-amenable group. Then there is a free measure preserving, ergodic, action of Γ on a standard Borel probability space such that E_Γ includes an equivalence relation with infinitely many ends.*

By theorem 4.4, we then obtain some $E \subset E_\Gamma$ such that E has cost greater than 1. Applying corollary 4.3 we can then get a free action of $\mathbb{F}_2$ on some non-null $A \subset X$ with $E_{\mathbb{F}_2} \subset E|_A$; in fact, the free group on two generators contains the free group on infinitely many generators, so we can as well get some $E_{\mathbb{F}_\infty} \subset E|_A$. With careful reorganization we can then get a free action of $\mathbb{F}_2$, spread out from A to the rest of the space, again with orbit equivalence relation included in $E \subset E_\Gamma$.

Corollary 4.6 *(Gaboriau-Lyons) Let Γ be a countable non-amenable group. Then there is a free measure preserving action of Γ on a standard Borel probability space (X, μ) and a free measure preserving action of $\mathbb{F}_2$ on (X, μ) with*

$$E_{\mathbb{F}_2} \subset E_\Gamma.$$

This is the corollary, which at the time seemed so innocuous, which underpins the final chapter of this story.

4.2 Open problems in the theory of cost and treeable equivalence relations

A quick rest at the side of the road.

Although it is not relevant to the main story, there are many open problems in the theory of cost. Most of these have been circulated in the literature for several years, and in some cases have been subjected to repeated attacks.

Definition For Γ a group, we define the *price* of Γ to be the infimum over all

$$C_\mu(E),$$

where E arises as the orbit equivalence relation induced by a free, measure preserving, ergodic action of Γ on a standard Borel probability space (X, μ). Γ is said to have *fixed price* if every free measure, preserving, ergodic action of Γ on a standard Borel probability space gives rise to an orbit equivalence relation with the same cost – namely, equal to that of the price of Γ.

Question Does every countable group have fixed price?

Definition A countable group Γ is said to have *property (T)* if it has property (T) over itself in the earlier sense – that is to say, whenever it acts unitarily on a Hilbert space with almost invariant vectors, there is a non-trivial Γ-invariant vector.

Question Does every property (T) group have price 1? For that matter, does every property (T) group have fixed price 1?

In general there are non-amenable groups with fixed price equal to one. [4] shows that whenever Γ and Δ are countable infinite groups, then $\Gamma \times \Delta$ is fixed price equal to 1. Thus, if one of the groups were to be non-amenable, which in turn implies $\Gamma \times \Delta$ non-amenable, we will have a non-amenable group with fixed price equal to one.

Definition Let E, F be countable Borel equivalence relations on a standard Borel space X. We say that E *has finite index in* F if $E \subset F$ and every F class contains only finitely many E classes. We say that E *has index* N *in* F if every F-class contains exactly N many E-classes.

Question If E and F are measure preserving, countable Borel equivalence relations on a standard Borel probability space (X, μ), and E has finite index in F, and E is treeable, must F be treeable a.e.?

Finally, an innocent looking question which has unexpectedly been shown by Abert and Nikolov to have important consequences in geometry:

Question Let Γ be a countable group and $\Delta \lhd \Gamma$ a normal subgroup of finite index N. Suppose Γ acts freely, ergodically, and in a measure preserving manner on a standard Borel probability space (X, μ), then do we have necessarily have

$$C_\mu(E_\Delta) = C_\mu(E_\Gamma) + \frac{N-1}{N}?$$

Indeed, one can put this in a more general setting, which is also open:

Question If E and F are measure preserving, countable Borel equivalence relations on a standard Borel probability space (X, μ) and E has index N in F, must we have

$$C_\mu(E) = C_\mu(F) + \frac{N-1}{N}?$$

4.3 Epstein's theorem

This result should be put in to proper context. It appeared about a year or so after [13] and certain aspects of the proof were inspired by the argument given there. I am going to organize the proof somewhat differently, admittedly taking numerous shortcuts, leaving key claims unproved, and making minor simplifying assumptions. The presentation I am giving will tend to obscure the links with [13].

Theorem 4.7 *(Epstein, [3]) Let Γ be a countable, non-amenable group. Then Γ has continuum many orbit inequivalent free, measure preserving, ergodic, actions on standard Borel probability spaces.*

Proof (Sketch only)

By Gaboriau-Lyons we can find a free, measure preserving, ergodic action of Γ on a standard Borel probability space (X, ν) which includes an equivalence relation $E_{\mathbb{F}_3}$ induced by a free measure preserving action of $\mathbb{F}_3$. For each t between 0 and 1, recall the action we recall the constructions given in the proof of Gaboriau-Popa and obtain an action of

$$\mathbb{F}_3 \cong \langle a, b, \varphi_t \rangle$$

on $\hat{\mathbb{Z}}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$.

Now let X_t be the collection of

$$\{(x, f) : x \in X, f : [x]_\Gamma \to \hat{\mathbb{Z}}^2, \forall \sigma \in \langle a, b, \varphi_t \rangle, \forall y \in [x]_\Gamma (f(\sigma \cdot y) = \sigma \cdot (f(y)))\}.$$

We let Γ act on X_t by

$$\gamma \cdot (x, f) = (\gamma \cdot x, f)$$

and we let $\langle a, b, \varphi_t \rangle$ act by

$$\sigma \cdot (x, f) = (\sigma \cdot x, f),$$

where here $\sigma \cdot x$ refers to the action of $\mathbb{F}_3 \cong \langle a, b, \varphi_t \rangle$ on X established as a result of Gaboriau-Lyons. We let F_t be the orbit equivalence relation of Γ on X_t and we let E_t be the orbit equivalence relation of $\langle a, b, \varphi_t \rangle$ on X.

There is only one natural Borel structure on X_t, though there are a few different ways this can be described. Let $\mathcal{B}_t$ be the σ-algebra on X_t generated by sets of the form

$$\{(x, f) : f(\gamma, x) \in B\}$$

for $\gamma \in \Gamma$ and $B \subset \hat{\mathbb{Z}}^2$ Borel. It can then be shown that $(X_t, \mathcal{B}_t)$ forms a standard Borel space.

The measure for the space X_t involves taking the product measure along the separate $\mathbb{F}_3$ orbits: Given $\{y_i : i \in \mathbb{N}\}$ a sequence of representatives of the orbits of $\langle a, b, \varphi_t \rangle$ included in $[x]_\Gamma$, we let

$$\mu_x^t = \prod_{\mathbb{N}} \mu$$

where the i^{th} copy of μ, our canonical measure on $\hat{\mathbb{Z}}^2$, is applied to y_i and obtain a measure on

$$X_{t,x} = \{(y, f) \in X_t : y = x\}.$$

Thus for $B_1, B_2, ..., B_N \subset \hat{\mathbb{Z}}^2$ Borel, $y_1, y_2, ..., y_N \in [x]_\Gamma$ with each $[y_i]_{E_t} \neq [y_j]_{E_t}$ for $i \neq j$, we have

$$\mu_x^t(\{(x, f) : \forall i \leq N(f(y_i) \in B_i)\}) = \prod_{i \leq N} \mu(B_i).$$

Using $\mathbb{F}_3$ invariance of μ, it turns out that this measure is independent of the choice of representatives $\{y_i : i \in \mathbb{N}\}$ we choose. From this it follows that if we define

$$m_t = \int_X \mu_x^t d\nu(x)$$

we obtain an E_Γ-*invariant* probability measure on X_t.

Thus we obtain an induced free, measure preserving action of Γ on X_t with

$$\gamma \cdot (x, f) = (\gamma \cdot x, f).$$

Let F_t denote the orbit equivalence relation arising from this action of Γ.

At this stage we do not know the new action of Γ is ergodic, but for conceptual simplicity, towards seeing the main idea, let us just assume the action is ergodic. (At this point Epstein took the ad hoc solution of going to an ergodic component – though later Ioana, Kechris, and Tsankov, in [14], performed a more general analysis of this peculiar kind of skew product action.)

For $\vec{a} \in \mathbb{Z}^2$ we can again define a function

$$\vec{a} : X_t \to \mathbb{C}$$

$$(x, f) \mapsto \vec{a}(f(x)),$$

where $\vec{a}(f(x))$ now carries the meaning for us it had during the proof of Gaboriau-Popa.

Claim: Each F_t is orbit equivalent to only countably many other F_s's.

Proof of Claim: Otherwise, adapting the earlier argument, we obtain $s_1 > s_2$ with

$$\theta : X_{s_1} \to X_{s_2}$$

witnessing orbit equivalence such that on a large finite set $F \subset \mathbb{F}_2 = \langle a, b \rangle$ and for a large measure of

$$(x, f) \in X_{s_1}$$

(a) $\theta(\gamma \cdot (x, f)) = \gamma \cdot \theta(x, f)$ all $\gamma \in F$;
(b) $\vec{a}(x, f) \sim \vec{a}(\theta(x, f))$ for $\vec{a}$ one of our two basis elements for $\mathbb{Z}^2$.

Note that I am taking $s_1 > s_2$ – this will make the argument slightly cleaner in a few steps time.

Arguing in analogy with the previous proof of Gaboriau-Popa, there is some positive measure

$$B \subset X_{s_1}$$

such that

$$\vec{a}(x, f) = \vec{a}(\theta(x, f))$$

for all $(x, f) \in B$.

Now we do not know that this yields $\theta(x, f) = (x, f)$, since $\mathbb{Z}^2$ is sufficient to separate points in $\hat{\mathbb{Z}^2}$ but retreats to agnosticism when faced with questions on X.

Subclaim: For m_{s_1} a.e. $(x, f) \in B$

$$\theta[[(x, f)]_{\langle a,b,\varphi_{s_1}\rangle}]$$

is included in a single $\langle a, b, \varphi_{s_1}\rangle$ orbit.

Proof of Subclaim: The nature of the measure m_{s_2} gives that for a.e. $(y, g) \in X_{s_2}$ the different E_{s_2}-equivalence classes included in $[y]_\Gamma$ are unrelated under the entire action of $\langle a, b, \varphi\rangle$; in particular, if $[y_1]_{\langle a,b,\varphi_{s_2}\rangle}$ and $[y_2]_{\langle a,b,\varphi_{s_2}\rangle}$ are distinct orbits with

$$y_1 E_\Gamma y_2,$$

then $g(y_1)$ is not an element of

$$[y_2]_{\langle a,b,\varphi_{s_1}\rangle}.$$

(□Subclaim)

We are still not done. Let us define

$$\hat{F} \subset F_{s_1}$$

by setting $(x, f)\hat{F}(y, g)$ if

$$(x, f)F_{s_1}(y, g)$$

and

$$f(x)E_{s_2}g(y).$$

It is clear that $\hat{F}$ has infinite index in F_{s_1}, but that does not suffice to finish the proof. It is the next subclaim which completes the task.

Subclaim: $\hat{F}|_B$ has index greater than 1 in $F_{s_1}|_B$.

Proof of Subclaim: I will assume otherwise and head for a contradiction.

For notational convenience, let us assume $\mu_{s_1}(B) > \frac{1}{2}$. In fact this is minor assumption, because an appropriately tight form of (a) and (b) will go through the Gaboriau-Popa argument and yield this conclusion.

Given the similarity the action of $\mathbb{F}_2 = \langle a, b\rangle$ on the spaces X_{s_1} and X_{s_2} we may assume without loss of generality we can now assume B is $\mathbb{F}_2$-invariant. It can be then shown that there exists some $C \subset B$ of positive measure such that:

(a) $\varphi_{s_1}|_C$ sends every point in C to an $\hat{F}$-*inequivalent* point.
(b) $\mathbb{F}_2 \cdot C = B$.

(a) in particular implies $\varphi_{s_1}(z) \notin B$ all $z \in B$.

For each $z \in B$ we choose $s(z) \in C \cap \mathbb{F}_2 \cdot z$ in a uniformly Borel manner. We can also then choose $\sigma_z \in \mathbb{F}_2$, again in a Borel manner, such that

$$\sigma_z \cdot z = s(z).$$

We then define

$$\rho : B \to X \setminus B$$

$$z \mapsto \sigma_z \cdot \varphi_{s_1}(\sigma_z \cdot z).$$

This is a Borel map included in the graph of F_{s_1} and hence is measure preserving. Now I claim it is one-to-one, which will give us a contradiction to $m_{s_1}(B) > \frac{1}{2}$.

The point is that if

$$\sigma_z \cdot \varphi_{s_1}(\sigma_z \cdot z) = \sigma_w \cdot \varphi_{s_1}(\sigma_w \cdot w)$$

for $z, w \in B$ then we would have that $z\hat{F}w$ by the assumption on B, which means they can be connected in the action of $\mathbb{F}_3 \cong \langle a, b, \varphi_{s_1}\rangle$ without passing through any applications of $\varphi_{s_1}|_C$.

Thus, since $\langle a, b, \varphi_{s_1}\rangle$ is acting freely, we have must have

$$\varphi_{s_1}(\sigma_z \cdot z) = \varphi_{s_1}(\sigma_w \cdot w),$$

which implies $\sigma_z = \sigma_w$ and then that $z = w$. (□Subclaim)

(□Claim)

□

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