Measure Theory, 2008, Homework Two

Tuesday, May 13, start of class

Q1: (2pts) Let $I = [0,1]$ be the closed unit interval in its usual compact topology and let $C(I)$ the collection of continuous functions on I with the metric $d(f,g) = \sup_{z \in I} |f(z) - g(z)|$.

Define

$$\Phi : C(I) \to C(I)$$

by

$$\Phi(f)(x) = \int_0^x f(x) d\lambda(x),$$

where λ is Lebesgue measure.

Show that Φ is a continuous function in the indicated topology on $C(I)$.

Q2:

Notation For K a compact metric space, let $P(K)$ be the probability measures on K equipped with the topology generated by the basic open sets

$$\{\mu : s_1 < \mu(f_1) < r_1, s_2 < \mu(f_2) < r_2, ..., s_n < \mu(f_n) < r_n\}$$

for $f_1, f_2, ..., f_n \in C(K)$.

(a) (2pts) Let K be a compact metric space. Let $C(K, [-1,1])$ be the subspace of $C(K)$ consisting of continuous functions with norm at most one – that is to say, the range included in $[1,-1]$ Show that if $\{f_i : i \in \mathbb{N}\}$ is a countable dense subset of $C(K, [-1,1])$ then the function

$$\pi : P(K) \to \prod_{i \in \mathbb{N}} [0,1]$$

given by

$$(\pi(\mu))(n) = \mu(f_n)$$

is continuous and open onto its image (i.e. π effects a homeomorphism between $P(K)$ and $\pi[P(K)]$).

(b) (2pts) Show that $P(K)$ is a compact metrizable space.

(c) (2pts) Let $\psi : K \to K$ be a homeomorphism. At each n let F_n be the collection of $\mu \in P(K)$ for which

$$|\int f d\mu - \int f \circ \psi d\mu| \leq \frac{1}{n}$$

for each $f \in C(K)$ with $-1 \leq f \leq f$.

Show that each F_n is a closed, non-empty subset of $P(K)$.[1]

(d) (2pts) Use (b) and (c) to conclude that there is a ψ-invariant Borel probability measure on K.

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[1]For non-empty: Given n, choose any $\nu \in P(K)$ and define $\Lambda \in C(K)$ by

$$f \mapsto \frac{1}{2n+1} \sum_{-n \leq \ell \leq n} \int f \circ \psi^\ell d\nu.$$

Observe that Λ is a positive, linear functional, with $\Lambda(1) = 1$, and then argue the measure corresponding to Λ is as required.