Measure Theory, 2008, Homework One

Due Date: Tuesday, April 22, start of class

Q1: (2pts) Let (X, Σ) be a measure space (i.e. Σ is a σ-algebra on X). Let μ be a measure on (X, Σ). Assume: (i) $\mu(X) = \infty$; (ii) μ is σ-finite; (iii) $\mu(\{x\}) = 0$ all $x \in X$ (that is to say, μ is atomless).

Show that we can write X as a disjoint union of sets

$$X = \dot{\bigcup}_{n\in\mathbb{N}} X_n$$

each with measure one.

Q2: (2pts) Let (X, Σ) be a measure space. Let $f : X \to \mathbb{R}$ be such that

$$f^{-1}[(-\infty, q)] \in \Sigma$$

all $q \in \mathbb{Q}$.

Show that f is measurable with respect to Σ (i.e. the pullback of any open set along f is in Σ).

Q3: (2pts) Let X be the closed unit square, $[0, 1] \times [0, 1]$ equipped with the subspace topology (from the usual topology on $\mathbb{R}^2$). Let Σ be the resulting σ-algebra of Borel subsets of X. Let μ be Lebesgue measure on X (i.e. the restriction of the measure m on $\mathbb{R}^2$ defined on page one of the course notes).

Let

$$f : X \to \mathbb{R}$$

be defined by

$$(x, y) \mapsto x^2y^2.$$

Let Σ_0 be the σ-algebra consisting of all sets of the form $A \times [0, 1]$ for $A \subset [0, 1]$ Borel.

Calculate $E(f|\Sigma_0)$, the conditional expectation of f with respect to Σ_0.

Q4: (4pts) Let

$$X = \prod_{n\in\mathbb{N}} \{0, 1\},$$

with the product topology. Let μ be the product measure on this space. (This is to say, for $A = \{f \in X : f(1) = \ell_1, f(2) = \ell_2, ..., f(n) = \ell_n\}$, we have $\mu(A) = 2^{-n}$.)

For each finite $S \subset \mathbb{N}$ define

$$\psi_S : X \to \mathbb{R}$$

$$f \mapsto (-1)^{-|\{n:f(n)=0\}|}.$$

Show that $\{\psi_S : S \subset \mathbb{N}, S \text{ finite}\}$ gives an orthonormal basis for the Hilbert space $L^2(X, \mu)$.[1]

I have put the current version of the course notes on line at:

http://www.math.ucla.edu/ ˜greg/measure.html

I will try to keep updating this site.

[1]Remarks: $\psi_\emptyset$ is the function with constant value 1. For orthoganility, try to show $\langle\psi_S, \psi_T\rangle = \int \psi_{S\Delta T} d\mu$. The final issue is to see that the linear combinations of these functions are dense in $L^2(X, \mu)$.