Treeable equivalence relations

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Abstract

There are continuum many $\leq_B$-incomparable equivalence relations induced by a free, Borel action of a countable non-abelian free group – and hence, there are $2^{\aleph_0}$ many treeable countable Borel equivalence relations which are incomparable in the ordering of Borel reducibility

1 Introduction

Definition An equivalence relation E on a standard Borel space X is said to be *Borel* if it is Borel as a subset of $X \times X$ – that is to say, it appears in the σ-algebra on $X \times X$ generated by rectangles of the form $A \times B$, A and B Borel subsets of X. The equivalence relation is said to be *countable* if every equivalence class is countable. It is said to be *treeable* if there is a Borel acyclic graph $\mathcal{G}$ whose connected components are the E-classes – in other words, there is a Borel forest such that $[x]_E = \{y \in X : yEx\}$ equals the connected subgraph of $\mathcal{G}$ containing x. In the case that X carries a standard Borel probability measure μ we say that E is *measure preserving* if for each Borel bijection

$$f : A \to B,$$

with A and B Borel sets, $f(x)Ex$ all $x \in A$, we have

$$\mu(A) = \mu(B).$$

We say that E is *ergodic* if any Borel E-invariant set is either null or conull.

Example Any free Borel action of the free group, $\mathbb{F}_2$, on a standard Borel space gives rise to a treeable countable Borel equivalence relation. Write $\mathbb{F}_2 = \langle a, b \rangle$, and connect $x\mathcal{G}y$ if for some $c \in \{a^{\pm 1}, b^{\pm 1}\}$ we have $c \cdot x = y$.

In the case that $\mathbb{F}_2$ acts in a measure preserving manner, the resulting equivalence relation is measure preserving.

One natural way in which such actions arise is from the shift action. Equip $2^{\mathbb{F}_2}$, i.e.

$$\prod_{\mathbb{F}_2} \{0, 1\},$$

with the product measure. There is an invariant, Borel, conull set $X \subset 2^{\mathbb{F}_2}$ on which $\mathbb{F}_2$ acts freely. The resulting equivalence relation is ergodic, measure preserving, Borel, and treeable.

Example Consider the linear action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^2$. The integer lattice $\mathbb{Z} \times \mathbb{Z}$ is invariant under this action, so we can pass to the quotient action of $\mathrm{SL}_2(\mathbb{Z})$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. There is a canonical Borel probability measure on this space, arising from the Lebesgue measure on the complete section $[0, 1) \times [0, 1)$ for the surjection

$$p : \mathbb{R}^2 \to \mathbb{T}^2.$$

The resulting action is measure preserving, ergodic, and free on a conull subset. It is well known, and can be found for instance in [15], that an isomorphic copy of $\mathbb{F}_2$ appears inside $\mathrm{SL}_2(\mathbb{Z})$, and hence we can obtain a free action of the free group.

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Notation For Γ a group and X a standard Borel space equipped with a Borel action of Γ,

$$a : \Gamma \times X \to X,$$

we use E_a to denote the orbit equivalence relation,

$$x_1 E_a x_2$$

if there exists some $\gamma \in \Gamma$ with $a(\gamma, x_1) = x_2$.

In the field of descriptive set theory the study of equivalence relations has been largely organized around the notion of Borel reducibility.

Definition For E and F equivalence relations on standard Borel spaces X and Y, we say that *E is Borel reducible to F,* written

$$E \leq_B F,$$

if there is a Borel function $\theta : X \to Y$ such that for all $x_1, x_2 \in X$,

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

Intuitively, $E \leq_B F$ can be thought of as asserting there is an "effective" or "reasonably concrete" injection

$$X/E \hookrightarrow Y/F.$$

In answer to a question considered in [8], [9], and [17], we show that among the treeable equivalence relations there are continuum many up to Borel reducibility.

Theorem 1.1 *There are countable Borel treeable equivalence relations, $(E_s)_{s\in\mathbb{R}}$, such that for $s \neq t$ we have E_s* not *Borel reducible to E_t.*

The proof gives some further information.

Theorem 1.2 *For $n \geq 2$, there are standard Borel probability spaces $(X_s, \mu_s)_{s\in\mathbb{R}}$, each equipped with a free measure preserving ergodic action of $\mathbb{F}_n$,*

$$a_s : \mathbb{F}_n \times X_s \to X_s,$$

such that for any $s \neq t$, μ_s-conull $A_s \subset X_s$, the restriction of E_{a_s} to A_s is not Borel reducible to E_{a_t}.

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3 Background and related results

Among the countable Borel equivalence relations, the *smooth* and *hyperfinite* equivalence relations can be thought of as the very simplest of their kind.

Definition An equivalence relation E on a standard Borel space X is said to be *smooth* if there is a Borel function

$$\theta : X \to Y,$$

from X to some other standard Borel space Y, such that

$$x_1 E x_2 \Leftrightarrow \theta(x_1) = \theta(x_2);$$

in other words, E is Borel reducible to the identity relation on Y.

In some sense the smooth Borel equivalence relations can be dismissed as the trivial cases. It is easily seen that any countable Borel equivalence relation has either countably many equivalence classes or continuum many. If E and F are smooth and have the same number of equivalence classes, then

$$E \sim_B F,$$

that is to say, each is Borel reducible to the other. The quotient objects arising from smooth countable equivalence relations can be identified with Borel subsets of standard Borel spaces, and hence provide no new phenomena not already found in classical descriptive set theory and the study of Borel sets and standard Borel spaces.

Definition An equivalence relation E on standard Borel X is *hyperfinite* if there is an increasing sequence of finite Borel equivalence relations,

$$F_0 \subset F_1 \subset ...F_n \subset ...$$

with each F_n Borel and having only finite equivalence classes, with

$$E = \bigcup_{n \in \mathbb{N}} \uparrow F_n.$$

Example The canonical example of a non-smooth hyperfinite equivalence is given by, E_0, eventual agreement on infinite binary sequences. Thus for $f, g \in 2^{\mathbb{N}}$, set

$$f E_0 g$$

if there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$f(n) = g(n).$$

The structure of hyperfinite equivalence relations was examined in [3]. In particular, drawing on earlier work of Ted Slaman, John Steel, and Benji Weiss:

Theorem 3.1 *(Dougherty-Jackson-Kechris, Weiss, Slaman-Steel; see[3]) An equivalence relation is hyperfinite if and only if it arises as the orbit equivalence relation induced by a Borel action of $\mathbb{Z}$.*

They also observe as part of this analysis that every hyperfinite equivalence relation is treeable and every smooth countable Borel equivalence relation is hyperfinite. Rephrasing, as do the authors of [11], a classical theorem into the language of Borel reducibility, we obtain that E_0 provides a *base* for the non-smooth countable Borel equivalence relations.

Theorem 3.2 *(Glimm, Effros) Let E be a countable Borel equivalence relation. Then either E is smooth or $E_0 \leq_B E$.*

Theorem 3.3 *(Dougherty-Jackson-Kechris, [3]) If E, F are countable Borel non-smooth hyperfinite equivalence relations, then*

$$E \sim_B F.$$

Thus the hyperfinite and smooth equivalence relations both have a completely transparent, and almost trivial, structure under $\leq_B$. There is a single example of a non-smooth hyperfinite equivalence relation up to Borel reducibility. There are exactly $\aleph_0$ many smooth countable Borel equivalence relations, and they can be classified depending on whether they have 1, 2, 3,..., $\aleph_0$, or $2^{\aleph_0}$ many equivalence classes.

For many years it was not even known how to produce $\leq_B$-incomparable countable Borel equivalence relations. This was finally settled at the end of the 90's by Scot Adams and Alexander Kechris.

Theorem 3.4 *(Adams-Kechris, [2]) There are continuum many $\leq_B$-incomparable countable Borel equivalence relations.*

The proof given there was based on the superrigidity and cocycle theory of Zimmer for algebraic groups. A more elementary proof, essentially free of black boxes, was presented in [9].

Yet in all these cases it was absolutely unclear how any kind of rigidity results could be adapted for treeable equivalence relations or free actions of a free group. Indeed, a careful inspection of the proofs shows the intrinsic role of *product group actions*. This is made explicit in [9]. It is more hidden in [2], where one has to go back to machinery of [18] – where the cocycle rigidity results for an algebraic group G of sufficiently high rank uses the existence of a subgroup of the form $H \times \Delta < G$ where neither group is compact and at least one is non-amenable.

In contrast to this, there has been a sequence of theorems, starting with a paper by Scot Adams, [1], which show these kinds of groups give rise to equivalence relations which are *not* treeable. For instance:

Theorem 3.5 *(Hjorth, [6]) Let G_1, G_2 be lcsc groups. Assume neither is compact and G_1 is non-amenable. Let $a : (G_1 \times G_2) \times X \to X$ be a measure preserving free action on a standard Borel probability space (X, μ).*

Then the result equivalence relation E_a is not Borel reducible to a treeable countable Borel equivalence relation.

Corollary 3.6 *Let G_1, G_2 be as above. Suppose $G_1 \times G_2$ is a closed subgroup of lcsc H. Let H act freely and by measure preserving transformations on a standard Borel probability spaces (X, μ),*

$$a : H \times X \to X.$$

Then the resulting orbit equivalence relation E_a is not Borel reducible to a treeable countable Borel equivalence relation.

Proof Suppose otherwise. Then by [11] we can assume that there is a free action b of $\mathbb{F}_2$ on standard Borel Y with a Borel function

$$\theta : X \to Y$$

witnessing

$$E_a \leq_B E_b.$$

Following [12] we can find $B \subset X$ Borel meeting each orbit in a countable set such that for any compact neighborhood $V \subset H$ of the identity and $x \in B$,

$$|V \cdot x \cap B| < \aleph_0.$$

Let

$$C = \theta[B].$$

Lusin-Novikov gives that C will be Borel and there will be some Borel

$$\rho : C \to B$$

with $\theta \circ \rho(y) = y$ all $y \in C$. Let $B_0 = \rho[C]$. Now the reduction on B_0 is one-to-one. B_0 is still Borel, still provides a complete section for the equivalence relation, and still has

$$|V \cdot x \cap B_0| < \aleph_0$$

all $x \in B_0$, all compact neighborhoods of the identity in H.

Fix compact neighborhood V of the identity in H. Again Lusin-Novikov gives

$$B_1 = V \cdot B_0$$

is Borel. Since V is non-null in H we have that B_1 will be non-null in X. To obtain a contradiction to 3.5 we only need to show that the orbit equivalence relation of $G_1 \times G_2$ on B_1 is treeable.

Let $\mathcal{F}(X)$ be the collection of all finite subsets of X; in the natural Borel structure, generated by sets of the form

$$\{F \in \mathcal{F}(X) : F \cap B \neq \emptyset\},$$

for $B \subset X$ Borel, we have that $\mathcal{F}(X)$ is a standard Borel space. (See for instance the discussion of the Effros standard Borel structure in [13].) Let Z be the collection of functions

$$f : \mathbb{F}_2 \to Y \times \mathcal{F}(X)$$

which are equivariant in the sense that

$$f(\sigma) = (y, F)$$

implies $f(\gamma\sigma) = (b(\gamma, y), F')$, some F'. We let $\mathbb{F}_2$ act on Z by shift on the right:

$$c(\gamma, f)(\sigma) = f(\sigma\gamma).$$

This action is free and Borel, hence E_c is treeable.

Applying [12] once more we obtain a complete section $C_0 \subset B_1$ for the action of $G_1 \times G_2$ with

$$(V \cap (G_1 \times G_2)) \cdot x \cap C_0$$

finite for each $x \in C_0$. Let $\phi : C_0 \to B_0$ be such that

$$\phi(x) \in V \cdot x$$

all x. Note that ϕ will be finite to one on each $G_1 \times G_2$ orbit.

We now define $\psi : C_0 \to Z$ by

$$\psi(x)(\gamma) = (y, F)$$

where $y = b(\gamma, \theta\phi(x))$ and $F = \{x' \in (G_1 \times G_2) \cdot x \cap C_0 : \theta\phi(x') = y\}$.

This gives the reduction of the orbit equivalence relation of $G_1 \times G_2$ on B_1 to the orbit equivalence relation induced by the free action of $\mathbb{F}_2$ on Z, and the desired contradiction to 3.5. □

Thus, while it seemed implausible that the treeable equivalence relations could have the kind of transparent structure one finds with the hyperfinite, the methods for demonstrating that were absent.

In some form parallel issues, and the unsuitability of the Zimmer type superrigidity, were felt in the area of *orbit equivalence.*

Definition Let Γ and Δ be countable groups. Two Borel, measure preserving actions $a : \Gamma_1 \times X \to X$ and $b : \Delta \times Y \to Y$ on standard Borel probability spaces (X, μ) and (Y, ν) are said to be *orbit equivalent* if there is a measure preserving bijection

$$\theta : X \to Y$$

with

$$[\theta(x)]_F = \{\theta(y) : yEx\}$$

for μ a.e $x \in X$.

After a sequence of earlier partial results the situation for free actions of $\mathbb{F}_2$ was only settled relatively recently, using a subtle idea of $\mathrm{SL}_2(\mathbb{Z})$'s canonical action of $\mathbb{T}^2$ having a kind of relative property (T):

Theorem 3.7 *(Gaboriau-Popa, [5]) There are continuum many orbit inequivalent free, ergodic, measure preserving actions of* $\mathbb{F}_2$ *on standard Borel probability spaces.*

It should be pointed out that under Borel reducibility there is a *maximum* treeable equivalence relation:

Definition Let $\mathbb{F}_2$ act on $2^{\mathbb{F}_2}$ by shift: $(\gamma \cdot f)(\sigma) = f(\gamma^{-1}\sigma)$. Let Z be the set of all $f \in 2^{\mathbb{F}_2}$ such that $\gamma \cdot f \neq f$ all $\gamma \in \mathbb{F}_2$, γ not equal to the identity. Let $E_{\infty\mathcal{T}}$ be the orbit equivalence relation of $\mathbb{F}_2$ on Z.

Theorem 3.8 *(Jackson-Kechris-Louveau, [11]) If E is a treeable countable Borel equivalence relation, then*

$$E \leq_B E_{\infty\mathcal{T}}.$$

Since $E_{\infty\mathcal{T}}$ is clearly treeable, and standard techniques show it to not be hyperfinite, the authors of [11] were prompted to ask:

Question (Jackson-Kechris-Louveau) Does there exist a treeable equivalence relation E with

$$E_0 <_B < E <_B E_{\infty\mathcal{T}}?$$

This question was also posed in [9].

Using a completely ad hoc technique, created simply for the problem, this was answered positively in [8]. Unfortunately the technique did nothing other than provide a narrow solution, displaying the existence of a single example between E_0 and $E_{\infty\mathcal{T}}$.

A scenario for at least showing the existence of countably many incomparable Borel equivalence relations was proposed Simon Thomas in [17]. His suggestion was to consider to the canonical action of $\mathrm{SL}_2(\mathbb{Z})$ on one dimensional subspaces of $\mathbb{Q}_p^2$ for various different primes p; these can be shown to all be treeable a.e. using that $\mathbb{F}_2$ sits as a finite index subgroup in $\mathrm{SL}_2(\mathbb{Z})$. In higher dimensions the superrigidity theory of [18] was extended in [17] to show $\leq_B$-incomparability of the actions of $\mathrm{SL}_n(\mathbb{Z})$ on $n-1$-dimensional subspaces of $\mathbb{Q}_p^n$ for different primes p. The techniques of [18] do not adapt readily to the context of treeable equivalence relations, and Thomas' conjecture remains open.

The proof we give here of continuum many treeable equivalence relations is much closer to the construction of [5], in that we form these as subequivalence relations of the orbit equivalence relation of $\mathrm{SL}_2(\mathbb{Z})$. However this proof is independent of their argument and does *not* appeal to the relative property (T) property which they use there. The argument here was strongly influenced by work of Adrian Ioana at [10]; as in his argument, we use a kind of reduction into the action of $\mathrm{SL}_2(\mathbb{Z})$ on projective space.

4 E_0-ergodicity and amenability

There are two structural facts used in the proof. First of all the amenability of $\mathrm{SL}_2(\mathbb{Z})$'s canonical action on projective space $P_1(\mathbb{R})$, the space of one dimensional subspaces of $\mathbb{R}^2$. Secondly the E_0-ergodicity of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$.

These can be thought of as contrasting. E_0-ergodicity is like a very strong failure of amenability.

The results in this section are all well known, though in many cases it is hard to say where they were first observed and whether they should be considered as folklore.

In most of what follows we will have to consider various different actions of the same group on the same space. I will usually take care to specifically indicate the action intended. For the case of $\mathrm{SL}_2(\mathbb{Z})$ with its usual linear action on $\mathbb{R}^2$ and $\mathbb{T}^2$ I will use the unadorned $\cdot$, so that these actions are being given a privileged notational status.

Notation For $M \in \mathrm{SL}_2(\mathbb{Z})$ and $\vec{x} \in \mathbb{R}^2$, we will always use

$$M \cdot \vec{x}$$

to denote the linear action of M on $\vec{x}$. Thus if $\vec{x} = (a, b)$ and $M = (a_{ij})_{i,j\in\{0,1\}}$, then

$$M \cdot \vec{x} = (a_{11}a + a_{12}b, a_{21}a + a_{22}b).$$

For

$$p : \mathbb{R}^2 \to \mathbb{T}^2$$
$$(a, b) \mapsto (a \bmod 1, b \bmod 1),$$

the canonical surjection, we we will let

$$M \cdot (p(\vec{x}))$$

denote $p(M \cdot \vec{x})$. Note that this action is well defined and Borel.

We let μ_2 denote the analog of Lebesgue measure for $\mathbb{T}^2$. Thus,

$$\mu_2(A) = \lambda_2(\{(x, y) \in [0, 1]^2 : p(x, y) \in A\},$$

where λ_2 is the usual Lebesgue measure on $\mathbb{R}^2$. Note that this measure is $\mathrm{SL}_2(\mathbb{Z})$-invariant.

Definition Let $P_1(\mathbb{R})$ be the collection of all one dimensional subspaces on $\mathbb{R}^2$. This is a standard Borel space in the Effros standard Borel structure – see for instance [13], §3.2 [11]. Consider the induced action of $\mathrm{SL}_2(\mathbb{Z})$, arising from its usual linear action on $\mathbb{R}^2$. Thus for $A \subset \mathbb{R}^2$, $M \in \mathrm{SL}_2(\mathbb{Z})$,

$$M \cdot A = \{M \cdot \vec{x} : \vec{x} \in A\}.$$

Lemma 4.1 *The induced orbit equivalence relation of* $\mathrm{SL}_2(\mathbb{Z})$ *acting on* $P_1(\mathbb{R})$ *is hyperfinite.*

See §3.2 of [11]

Lemma 4.2 *In the above described action of* $\mathrm{SL}_2(\mathbb{Z})$ *acting on* $P_1(\mathbb{R})$*, the stabilizer of any line is amenable.*

Proof It suffices to see that in the induced action of $\mathrm{GL}_2(\mathbb{R})$, the general linear group over all the reals, the stabilizer of any point is amenable. Since this action is now transitive, the stabilizer of the various lines are all isomorphic – hence one is amenable if and only if all are amenable.

Consider then the line consisting of all $\{(a, 0) : a \in \mathbb{R}\}$. The stabilizer of this line is the collection of upper triangular matrices, and hence not only amenable but solvable. □

Lemma 4.3 *Let* Γ *be a countable amenable group acting in a Borel manner on a standard Borel space* X*. Let* μ *be a standard Borel probability measure on* X*. Then there is a* μ*-conull set on which the induced orbit equivalence relation is hyperfinite.*

Proof Appealing to 2.13 [11] we obtain the orbit equivalence relation is 1-amenable. In particular it is μ-amenable. Now the result follows from the form of Connes-Feldman-Weiss presented at 2.6 [11]. □

Definition Let E be a countable, Borel, measure preserving equivalence relation on a standard Borel probability space (X, μ). Let $[E]$ denote the full group of E – that is to say, the set of Borel bijections

$$f : X \to X$$

with $f(x)Ex$ all $x \in X$.

E is said to have *almost invariant sets of measure 1/2* if there exists a sequence of measurable sets, $(A_n)_{n\in\mathbb{N}}$, with

$$\mu(A_n) \to \frac{1}{2}$$

and

$$\mu(\theta[A_n]\Delta A_n) \to 0$$

all $\theta \in [E]$.

E is said to be E_0*-ergodic* if whenever

$$\rho : X \to 2^{\mathbb{N}}$$

with $x_1 E x_2 \Rightarrow \rho(x_1) E_0 \rho(x_2)$ all $x_1, x_2 \in X$,

$$[\rho(x)]_{E_0}$$

is constant on a conull set.

Lemma 4.4 *Let* E *be a countable, measure preserving, ergodic Borel equivalence relation on a standard Borel probability space* (X, μ)*. Then the following are equivalent:*

(i) E *is* E_0*-ergodic;*

(ii) E *does not have almost invariant sets of measure 1/2;*

Moreover in the case that $E = E_\Gamma$ *is induced by the Borel action of a countable group* Γ *on* X*, these are both equivalent to:*

(iii) there does not exist a sequence of measurable sets, $(A_n)_{n\in\mathbb{N}}$*, with*

$$\mu(A_n) \to \frac{1}{2}$$

and

$$\mu(\gamma \cdot A_n \Delta A_n) \to 0$$

all $\gamma \in \Gamma$*.*

See the appendix of [9].

Definition A countable group Γ acting by unitary transformations on a Hilbert space $\mathcal{H}$ is said to have *almost invariant unit vectors* if for all finite $F \subset \Gamma$ and $\epsilon > 0$ there exists $v \in \mathcal{H}$, $\|v\| = 1$, with

$$\|\gamma \cdot v - v\| < \epsilon$$

all $\gamma \in F$. It is said to have an $F - \epsilon$*-invariant unit vector* if we witness the above just for the finite $F \subset \Gamma$ and positive ϵ – that is to say, there exists $v \in \mathcal{H}$, $\|v\| = 1$, with

$$\|\gamma \cdot v - v\| < \epsilon$$

all $\gamma \in F$.

Lemma 4.5 *Let Γ be a countable non-amenable group acting on a countable set S with every point having amenable stabilizer. Then the induced action on $\ell^2(S)$ does not have almost invariant vectors.*

See for instance [16] or [14].

Corollary 4.6 *Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be non-amenable. Then the induced action on*

$$L_0^2(\mathbb{T}^2, \mu_2) =_{df} \{f \in L^2(\mathbb{T}^2, \mu_2) : \int f d\mu = 0\}$$

does not have almost invariant vectors.

Proof The dual group of $\mathbb{T}^2$ is $\mathbb{Z}^2$ and the action of $\mathrm{SL}_2(\mathbb{Z})$ on $L_0^2(\mathbb{T}^2, \mu_2)$ is naturally isomorphic to the linear action of $\mathrm{SL}_2(\mathbb{Z})$ on $\ell^2(\mathbb{Z}^2 \setminus \{\vec{0}\})$. The stabilizer of any point in the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2 - \{\vec{0}\}$ is amenable, and now the result follows by the previous lemma. □

Corollary 4.7 *The action of any such non-amenable Γ on $\mathbb{T}^2$ does not admit almost invariant vectors of measure 1/2.*

There are two other minor, unrelated, but well known, facts we will need.

Definition For (X, μ) a standard Borel probability space, we let $M_\infty(X, \mu)$ be the collection of all measurable functions

$$\phi : X \to X$$

such that ϕ is measure preserving and one-to-one almost everywhere, with the identification of functions agreeing on a conull set. We equip this set with the topology of sub-basic open sets of the form

$$\{\phi \in M_\infty(X, \mu) : \mu(A \Delta \phi(B)) < \epsilon\}.$$

Lemma 4.8 *$M_\infty(X, \mu)$ is a Polish group.*

See for instance [13].

Lemma 4.9 $\mathrm{SL}_2(\mathbb{Z})$ *contains a copy of* $\mathbb{F}_2$.

See for instance [15] or [5].

Note then that since $\mathbb{F}_2$ contains a copies of free groups with any countable number of generators, we obtain $\mathbb{F}_n$ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ for any n.

5 Proof

Notation Let E denote the orbit equivalence relation induced by the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$.

Fix matrices $a, b, c \in \mathrm{SL}_2(\mathbb{Z})$ such that $\langle a, b, c\rangle$ is canonically isomorphic to $\mathbb{F}_3$. Let $E_{\mathbb{F}_3}$ denote the orbit equivalence relation induced by the action of $\langle a, b, c\rangle$.

Lemma 5.1 *This action of $\mathbb{F}_3$ on $\mathbb{T}^2$ is E_0-ergodic. In fact, the action of just $\langle b, c\rangle$ is E_0-ergodic.*

Proof By 4.7. □

Now, following a parallel step in [5] we will define a parameterized collection of morphisms, $(\phi_t)_{t\in\mathbb{R}}$ such that for $s < t$, and for a non-null $x \in \mathbb{T}^2$, $\{\phi_s^\ell(x) : \ell \in \mathbb{Z}\}$ is strictly included in $\{\phi_t^\ell(x) : \ell \in \mathbb{Z}\}$.

First fix a measurable partition $(A_n)_{n\in\mathbb{N}}$ of $\mathbb{T}^2$ with each $\mu_2(A_n) > 0$ and $\{a^\ell \cdot A_n : \ell \in \mathbb{Z}\} = \mathbb{T}^2$. Let $\{q_n : n \in \mathbb{N}\}$ be an enumeration of the rationals. For each $t \in \mathbb{R}$ let

$$B_t = \bigcup\{A_n : q_n < t\}.$$

Note then that for $s < t$ we have $B_s \subset B_t$ and $B_t \setminus B_s$ non-null.

Given $t \in \mathbb{R}$ we define $N_t : \mathbb{T}^2 \to \mathbb{N}$ by cases:

1. If $x \in B_t$ then $N_t(x)$ is the least $N > 0$ such that
$$a^N \cdot x \in B_t;$$
2. if $x \in A_n$ for some n with $q_n \geq t$, then $N_t(x)$ is the least $N > 0$ such that
$$a^N \cdot x \in B_n.$$

By the Poincare recurrence theorem, N_t is defined a.e. We then define

$$\phi_t : \mathbb{T}^2 \to \mathbb{T}^2$$
$$x \mapsto a^{N_t(x)} \cdot x.$$

ϕ_t is defined a.e. and the assignment

$$\mathbb{R} \to M_\infty(\mathbb{T}^2, \mu_2),$$
$$t \mapsto \phi_t$$

is measurable.

We then let $a_t : \mathbb{F}_3 \times \mathbb{T}^2 \to \mathbb{T}^2$ be the action defined by

$$a_t(a, x) = \phi_t(x),$$
$$a_t(b, x) = a(b, x),$$
$$a_t(c, x) = a(c, x).$$

(Strictly speaking this action is only defined on a conull subset of $\mathbb{T}^2$, but that is sufficient for our purposes.)

Then

$$\mathbb{R} \mapsto M_\infty(\mathbb{T}^2, \mu_2)$$
$$t \mapsto (x \mapsto a_t(\gamma, x))$$

will be Borel for any $\gamma \in \mathbb{F}_3$.

Lemma 5.2 *If $s < t$ then for a.e. x*

$$\{\phi_s^\ell : \ell \in \mathbb{Z}\} \subset \{\phi_t^\ell(x) : \ell \in \mathbb{Z}\};$$

moreover for a.e. $x \in A_t \setminus A_s$ this inclusion is strict.

Proof It suffices to consider the situation when $x \in A_t$, since in the other cases the orbits are immediately equal. Then the structure of the definition gives

$$\{\phi_t^\ell(x) : \ell \in \mathbb{Z}\} = \{a^\ell \cdot x : \ell \in \mathbb{Z}\} \cap A_t.$$

If $x \in B_m \cap A_t$ for some $q_m \geq s$, then

$$\{\phi_s^\ell(x) : \ell \in \mathbb{Z}\} = \{a^\ell \cdot x : \ell \in \mathbb{Z}\} \cap B_m,$$

which is strictly included in $\{a^\ell \cdot x : \ell \in \mathbb{Z}\} \cap A_t$. □

Lemma 5.3 *$[E_s : E_t] = \infty$ all $s < t$ – that is to say, each E_t class contains infinitely many E_s classes a.e.*

Proof Let C be the set of y for which $\{\phi_s^\ell(y) : \ell \in \mathbb{Z}\}$ is strictly included in $\{\phi_t^\ell(y) : \ell \in \mathbb{Z}\}$. The saturation of C under b and c will be conull by 5.1. Then for a.e. x we can find a infinite set $\{y_n : n \in \mathbb{N}\} \subset C$ included in the orbit under $\langle b, c\rangle$ of x.

At each y_n we can fix a

$$z_n \in \{\phi_t^\ell(y) : \ell \in \mathbb{Z}\} \setminus \{\phi_s^\ell(y) : \ell \in \mathbb{Z}\}.$$

Given that the original action of $\langle a, b, c\rangle$ on $\mathbb{T}^2$ was free a.e., we will have a.e. that no z_n is E_s-equivalent to z_m for $n \neq m$. □

Lemma 5.4 *E_s will be E_0-ergodic for all s.*

Proof By 5.1. □

Theorem 5.5 *For each $t_0 \in \mathbb{R}$, there are only countably many $s \in \mathbb{R}$ with*

$$E_s|_B \leq_B E_{t_0}$$

on some μ_2-conull $B \subset \mathbb{T}^2$.

Proof Assume instead there is an *uncountable* $A \subset \mathbb{R}$ with corresponding conull $(B_s)_{s\in A}$ having $E_s|_{B_s} \leq_B E_{t_0}$. For each $s \in A$ let

$$\rho_s : B_s \to \mathbb{T}^2$$

be a Borel reduction of the equivalence relations. Let

$$\alpha_s : B_s \times \mathbb{F}_3 \to \mathrm{SL}_2(\mathbb{Z})$$

be the cocycle given by the requirement

$$\alpha_s(x, \gamma) \cdot \rho_s(x) = \rho_s(a_s(\gamma, x)).$$

Let d_0 be the canonical complete compatible metric on $\mathbb{T}^2$:

$$d_0(x, y) = \inf\{|v_1 - u_1| + |v_2 - u_2| : (u_1, u_2), (v_1, v_2) \in \mathbb{R}^2, p(u_1, u_2) = x, p(v_1, v_2) = y\}.$$

For ρ_1, ρ_2 measurable functions from $\mathbb{T}^2$ to $\mathbb{T}^2$ we let

$$d_1(\rho_1, \rho_2) = \int d_0(\rho_1(x), \rho_2(x)) d\mu_2(x).$$

This gives a complete metric on this space of functions, where we identify functions agreeing a.e. The induced topology is separable.

Now fix an enumeration of $(\gamma_n)_{n\in\mathbb{N}}$ of $\mathbb{F}_3$. Given two cocycles

$$\alpha, \beta : \mathbb{F}_3 \times \mathbb{T}^2 \to \mathrm{SL}_2(\mathbb{Z})$$

we let

$$d_2(\alpha, \beta) = \sum_{n\in\mathbb{N}} 2^{-n} \mu_2(\{x : \alpha(\gamma_n, x) \neq \beta(\gamma_n, x)\}).$$

Again, assuming we identify cocyles agreeing a.e., this gives a complete second countable metric.

Claim: For $s_1 \neq s_2$,

$$\mu_2(\{x : \rho_{s_1}(x) = \rho_{s_2}(x)\}) = 0.$$

Proof of Claim: Suppose instead on some positive measure B we have $\rho_{s_1}(x) = \rho_{s_2}(x)$ all $x \in B$, with $s_1 < s_2$. Then we get

$$x E_{s_1} y \Leftrightarrow x E_{s_2} y$$

all $x, y \in B$.

Using $[E_{s_1} : E_{s_2}] = \infty$ we can find a non-null $C \subset A_{s_1} \cap A_{s_2}$ and a Borel injection

$$\theta : C \to \mathbb{T}^2$$

such that

$$\theta(x) E_{s_2} x,$$
$$\theta(x) \not\!E_{s_1} x,$$

all $x \in C$. By ergodicity of E_{s_1} we have

$$[x]_{E_{s_1}} \cap B \neq \emptyset$$

for a.e. x. Similarly

$$[x]_{E_{s_1}} \cap \theta[B] \neq \emptyset$$

for a.e. x. Choosing an x with both these properties we obtain a contradiction to our assumptions on θ. (□Claim)

Using second countability of our metrics we can find some $u \in A$ and $(s_n)_{n \in \mathbb{N}}$ with

$$d_1(\rho_{s_n}, \rho_u), d_2(\alpha_{s_n}, \alpha_u) \to 0,$$

and for all $\gamma \in \mathbb{F}_3$, the function

$$x \mapsto a_{s_n}(\gamma, x)$$

converges to

$$x \mapsto a_u(\gamma, x)$$

in the uniform topology on $[E]$ – that is to say,

$$\mu_2(\{x : a_{s_n}(\gamma, x) \neq a_u(\gamma, x)\}) \to 0.$$

Notation For $z \in B_u$ let $N(z)$ be

$$\sup\{\|\alpha_u(z, d)\| : d \in \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}\},$$

where $\|M\|$ of an element of $\mathrm{SL}_2(\mathbb{Z})$ refers to the matrix norm given by the sum of the absolute values of its entries.

Definition At each n define the relation R_n by

$$x R_n y$$

if and only if the following all take place:
(i) $x = a_u(y, d) = a_{s_n}(y, c)$ some $d \in \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$;
(ii) $\alpha_u(y, d) = \alpha_{s_n}(y, d)$;
(iii) $x, y \in B_u \cap \bigcap_{m \in N} B_{s_m}$;
(iv) $d_0(\rho_{s_n}(x), \rho_u(x)) < \frac{1}{100N(x)}$, $d_0(\rho_{s_n}(y), \rho_u(y)) < \frac{1}{100N(y)}$.

Let $\hat{E}_n$ be the equivalence relation arising from the transitization of R_n. For each $\theta \in [E_u]$,

$$\mu_2(\{x : x\hat{E}_n\theta(x)\}) \to 1.$$

Hence after possibly thinning out the sequence $(s_n)_{n \in \mathbb{N}}$ we can assume that for each $\gamma \in \mathbb{F}_3$ there is conull $B_\gamma \subset B_u \cap \bigcap_{n \in N} B_{s_n}$ such that for all $x \in B_\gamma$

$$\exists N \forall m > N(x\hat{E}_m a_u(\gamma, x)).$$

Let $F_n = \bigcap_{m > n} \hat{E}_m$. Then, on a conull set, E_u is the increasing union of the F_n's. In particular, we obtain at large enough N that F_N does not have almost invariant sets of measure $1/2$.

Fix this N. Let

$$B = \{x \in \mathbb{T}^2 : d_0(\rho_{s_N}(x), \rho_u(x)) < \frac{1}{100N(x)}\}.$$

F_N does not have almost invariant sets of measure $1/2$ and is trivial outside B, and hence there is some $C \subset B$ which is non-null and has $F_N|_C$ E_0-ergodic. After throwing away a null set we may assume $\rho_u(x) \neq \rho_{s_N}(x)$ all $x \in C$. The definition of B gives for each $x \in B$ a unique $(s,t) \in \mathbb{R}^2$, with

$$|s|, |t| < \frac{1}{100N(x)}$$

and

$$p(s,t) = \rho_u(x) - \rho_{s_N}(x),$$

for $p : \mathbb{R}^2 \to \mathbb{T}^2$ the canonical surjection. Denote this by $(s,t) = \psi(x)$.

Now let $f : \mathbb{R}^2 \setminus \{\vec{0}\} \to P_1(\mathbb{R})$ be the surjection which sends each (s,t) to the line passing through $\vec{0}$ and (s,t). Then define

$$\varphi : C \to P_1(\mathbb{R}),$$
$$x \mapsto f(\psi(x)).$$

Notation Let $E^{P_1(\mathbb{R})}_{\mathrm{SL}_2(\mathbb{Z})}$ denote the orbit equivalence relation given by the linear action of $\mathrm{SL}_2(\mathbb{Z})$ on $P_1(\mathbb{R})$.

Claim: If xR_Ny then $\varphi(x)E^{P_1(\mathbb{R})}_{\mathrm{SL}_2(\mathbb{Z})}\varphi(y)$.

Proof of Claim: Say $d \in \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$ with

$$y = a_u(x, d) = a_{s_N}(x, d).$$

Let $M \in \mathrm{SL}_2(\mathbb{Z})$ with

$$\alpha_u(x, d) = \alpha_{s_N}(x, d) = M.$$

By the definition of the cocycles this gives

$$M \cdot \rho_u(x) = \rho_u(y),$$

$$M \cdot \rho_{s_N}(x) = \rho_{s_N}(y).$$

Suppose $(u, v) \in \mathbb{R}^2$ with

$$\rho_u(x) = p(u, v).$$

Then we have $(s, t) \in \mathbb{R}^2$ with

$$|s|, |t| < \frac{1}{100N(x)}$$

and

$$\rho_{s_N}(x) = p(u + s, v + t).$$

$$\rho_u(y) = M \cdot \rho_u(x) = p(M \cdot (u, v))$$

while

$$\rho_{s_N}(y) = M \cdot \rho_{s_N}(x) = p(M \cdot (u + s, v + t)) = p(M \cdot (u, v)) + p(M \cdot (s, t)).$$

But the assumption $|s|, |t| < \frac{1}{100N(x)}$ yields that $M \cdot (s, t) = (s', t')$ for some s', t' with

$$|s'|, |t'| < \frac{1}{10}.$$

This in turn gives $\psi(y) = (s', t')$, and hence $\varphi(y) = M \cdot (s, t) = M \cdot \varphi(x)$. (□Claim)

Claim: If $\gamma \in \mathbb{F}_3$, xF_Ny, then $M \cdot \varphi(x) = \varphi(y)$.

Proof of Claim: Since the action of $\mathbb{F}_3$ is free, there will be a unique redundancy free sequence $d_0, d_1, ...d_k \in \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$ such that

$$y = a_u(d_0 d_1 ... d_k, x) = a_{s_N}(d_0 d_1 ... d_k, x).$$

The assumption of F_N equivalence give that at each $i < k$

$$a_u(d_0 ... d_i, x) R_N a_u(d_0 ... d_i d_{i+1}, x),$$

and now the result follows from the preceding claim. (□Claim)

Claim: $[\varphi(x)]_{E^{P_1(\mathbb{R})}_{\mathrm{SL}_2(\mathbb{Z})}}$ is constant for a.e. $x \in C$.

Proof of Claim: Since $F_N|_C$ is E_0-ergodic and $E^{P_1(\mathbb{R})}_{\mathrm{SL}_2(\mathbb{Z})}$ is hyperfinite. (□Claim)

Thus we can get some $D \subset C$, $\mu_2(D) > 0$ and $z \in P_1(\mathbb{R})$ such that

$$\varphi(x) = z$$

all $x \in D$. Let $\Delta < \mathrm{SL}_2(\mathbb{Z})$ be the stabilizer of z in the linear action on $P_1(\mathbb{R})$. This will be an amenable subgroup, and for all $x \in D, \gamma \in \mathbb{F}_3$, if $a_u(\gamma, x) = a_{s_N}(\gamma, x) \in D$ with $xF_N a_u(\gamma, x)$, then

$$\alpha_u(\gamma, x) \in \Delta.$$

Let D^+ be the saturation of $\rho_{s_n}[D]$ under the action of Δ. Let E_Δ be the orbit equivalence relation induced by Δ. Let $\nu = \rho^*_{s_n}(\mu_2)$ – that is to say, define ν by

$$\nu[B] = \mu_2(\rho^{-1}_{s_n}[B]).$$

Applying 4.3, we obtain that E_Δ is hyperfinite ν a.e. in D^+. Thus we can assume, without any damage to the generality of the argument, that E_Δ is hyperfinite.

Note here that for all $x, y \in D$

$$xF_N y \Rightarrow \rho_{s_n}(x) E_\Delta \rho_{s_n}(y).$$

Then E_0 ergodicity of $F_N|_D$ implies that ρ_{s_n} is constant on a positive measure subset of D – which contradicts our assumption that ρ_{s_n} witnessed $E_{s_n}|_{B_{s_n}} \leq_B E_{t_0}$. □

Corollary 5.6 *There is a perfect set $P \subset \mathbb{R}$ such that for any $s, t \in P$ and $B \subset \mathbb{T}^2$ conull, $E_s|_B$ is not Borel reducible to E_t.*

Proof Recall that the class of Borel sets is closed under the measure quantifier $\forall^\mu$ for any Borel probability measure μ. (See [13]). Thus to say the set of triples $(\rho, s, t) \in M_\infty(\mathbb{T}^2, \mu_2) \times \mathbb{R} \times \mathbb{R}$ such that $\rho : \mathbb{T}^2 \to \mathbb{T}^2$ Borel reduces E_s to E_t on some conull set is Borel.

Thus the set of pairs (s, t) for which such a ρ exists is $\underset{\sim}{\Sigma}^1_1$, or *analytic*. In particular it has the property of Baire.

However given that it has countable sections, it must be meager as a subset of $\mathbb{R} \times \mathbb{R}$. Hence we can find a perfect set $P \subset \mathbb{R}$ such that for all $s \neq t$ in P, E_s is not Borel reducible to E_t on any conull set. □

Theorem 5.7 *There exist $2^{\aleph_0}$ many treeable countable Borel equivalence relations which are pairwise incomparable under $\leq_B$.*

Proof Take $(E_s)_{s \in P}$ from the last result. □

Theorem 5.8 *For $n \geq 2$ there are $2^{\aleph_0}$ many free, measure preserving, Borel, ergodic actions of $\mathbb{F}_n$ on standard Borel probability spaces, $(a_\alpha)_{\alpha < 2^{\aleph_0}}$, each*

$$a_\alpha : \mathbb{F}_n \times X_\alpha \to X_\alpha,$$

with resulting orbit equivalence relations E_α on X_α, such that for $\alpha \neq \beta$, there is no conull set $A \subset X_\alpha$ with

$$E_\alpha|_A \leq_B E_\beta.$$

Proof The case for $n = 3$ is given by the argument above.

First consider the case for $n = 2$. Begin with the orbit equivalence relations E_α on standard Borel probability spaces, each E_α treeable, ergodic, induced by a free Borel action of $\mathbb{F}_3$, on standard Borel probability space (X_α, μ_α), with no E_α Borel reducible to another E_β on a conull set.

Replace each X_α with the disjoint union, of two copies,

$$Y_\alpha = X_\alpha \times \{0, 1\}.$$

Take the product measure, ν_α. Define F_α by

$$(x_1, i_1) F_\alpha (x_2, i_2)$$

if $x_1 E_\alpha x_2$. It follows from the induction theorem, or cost restriction formula, or [4] that the cost of F_α equals

$$C_{\nu_\alpha}(F_\alpha|_{X_\alpha \times \{0\}}) + \nu_\alpha(X_\alpha \times \{1\}),$$

which in turn equals

$$\frac{1}{2} C_{\mu_\alpha}(E_\alpha) + \frac{1}{2}.$$

A free action measure preserving action of a free group on a finite measure spaces is equal to the measure of the space multiplied by the number of generators, from [4]. Hence

$$\frac{1}{2} C_{\mu_\alpha}(E_\alpha) + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2.$$

By [7] we can find a free measure preserving action of $\mathbb{F}_2$ on Y_α which induces F_α on a conull set. Since E_α is orbit equivalent to $F_\alpha|_{X_\alpha \times \{0\}}$, this is as required.

For $n > 3$ the same kind of argument works, but now restricting the space to drive up the cost.

Take $A_\alpha \subset X_\alpha$ with measure equal to

$$1 - \frac{n-3}{n-1} = \frac{2}{n-1}.$$

Then the cost of $E_\alpha|_{A_\alpha}$ equals $3 - \frac{n-3}{n-1}$, again by Gaboriau's cost restriction formula from [4]. Renormalizing the measure with

$$\nu_\alpha = \mu_\alpha \cdot \frac{n-1}{2}$$

we obtain

$$F_\alpha = E_\alpha|_{A_\alpha}$$

with respect to μ_α equals

$$(3 - \frac{n-3}{n-1}) \cdot \frac{n-1}{2} = \frac{3n-3-n+3}{n-1} \cdot \frac{n-1}{2} = \frac{2n}{2} = n.$$

The subequivalence relation of a treeable equivalence relation is treeable, by [11], hence F_α is treeable. Then apply [7] to get a free action of $\mathbb{F}_n$ inducing F_α on a conull set. □

References

[1] S. Adams, *Indecomposability of tree equivalence relations,* **Israel Journal of Mathematics,** vol. 64 (1988), pp. 362-380.

[2] S. Adams, A.S. Kechris, *Linear algebraic groups and countable Borel equivalence relations,* **Journal of the American Mathematical Society,** vol. 10 (1990), pp. 903-943.

[3] , R. Dougherty, S. Jackson, A.S. Kechris, *The structure of hyperfinite equivalence relations,* **Transactions of the American Mathematical Society,** vol. 341 (1994), pp. 193-225.

[4] D. Gaboriau, *Cot des relations dé quivalence et des groupes,* (French. English summary) [Cost of equivalence relations and of groups] **Inventiones Mathematicae,**139 (2000), no. 1, 41–98.

[5] D. Gaboriau, S. Popa, *An uncountable family of nonorbit equivalent actions of F_n,* **Journal of the American Mathematical Society,** vol 18 (2005), pp. 547559.

[6] G. Hjorth, *Non-treeability for product group actions,* **Israel Journal of Mathematics,** vol. 163, (2008), pp. 383-409.

[7] G. Hjorth, *A lemma for cost attained,* **Annals of Pure and Applied Logic,** 143 (2006), no. 1-3, 87–102.

[8] G. Hjorth, *A converse to Dye's theorem,* **Transactions of the American Mathematical Society,** vol. 357 (2005), pp. 3083-3103.

[9] G. Hjorth, A.S. Kechris, *Rigidity theorems for actions of product groups and countable Borel equivalence relations,* **Memoirs of the American Mathematical Society,** vol. 177 (2005), no. 833.

[10] A. Ioana, *Relative property (T) for the subequivalence relations induced by the action of $SL_2(\mathbb{Z})$ on $\mathbb{T}^2$,* preprint.

[11] S. Jackson, A.S. Kechris, A. Louveau, *Countable Borel equivalence relations,* **Journal of Mathematical Logic,** vol. 2 (2002), pp. 1 - 80

[12] A.S. Kechris, *Countable sections for locally compact group actions,* **Ergodic Theory and Dynamical Systems,** vol. 12 (1992), pp. 283-295.

[13] A. S. Kechris, **Classical Descriptive Set Theory**, Springer, 1995, New York.

[14] A. S. Kechris, T. Tsankov, *Amenable actions and almost invariant sets,* **Proceedings of the American Mathematical Society,** vol. 136 (2008), pp. 687-697.

[15] W. Magnus, A. Karrass, D. Solitar, **Combinatorial group theory,** 1966, Interscience Publishers, New York.

[16] S. Popa, *Some computations of 1-cohomology groups and construction of non orbit equivalent actions,* preprint.

[17] S. Thomas, *Superrigidity and countable Borel equivalence relations*, **Annals of Pure and Applied Logic** vol. 120 (2003), pp. 237-262

[18] R.J. Zimmer, **Ergodic Theory and Semisimple groups,** Birkhäuser, 1994, Basel.

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