Independently Axiomatizable $\mathcal{L}_{\omega_1,\omega}$ Theories

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Abstract

In partial answer to a question posed by Arnie Miller [5] and X. Caicedo [2] we obtain sufficient conditions for an $\mathcal{L}_{\omega_1,\omega}$ theory to have an independent axiomatization. As a consequence we obtain two corollaries: If Vaught's conjecture holds, then every $\mathcal{L}_{\omega_1,\omega}$ theory in a countable language has an independent axiomatization; every intersection of a family of Borel sets can be formed as the intersection of a family of *independent* Borel sets.

1 Introduction

Definition 1. *A set of sentences T' is called* independent *if for every $\phi \in T'$, $T' \setminus \{\phi\} \nvDash \phi$.*

A theory T is called independently axiomatizable*, if there is a set T' which is independent and T and T' have exactly the same models.*

Note that this definition applies to sets of sentences in both first-order ($\mathcal{L}_{\omega,\omega}$) and infinitary ($\mathcal{L}_{\omega_1,\omega}$) logic, granted that we have defined a meaning for $\models$. The question is whether every theory has an independent axiomatization. For first-order theories the answer is positive:

Theorem 2. *(M.I. Reznikoff- [3]) All theories of any cardinality in $\mathcal{L}_{\omega,\omega}$, are independently axiomatizable.*

Definition 3. *For a set of sentences $T \subset \mathcal{L}_{\omega_1,\omega}$ and a sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$, write*

$$T \models \sigma,$$

if all the models of T satisfy σ.

Two sets of sentences in $\mathcal{L}_{\omega_1,\omega}$ are semantically equivalent *if they have exactly the same models.*

Using definition 1 and 3, it makes sense to ask whether a theory in $\mathcal{L}_{\omega_1,\omega}$ is independently axiomatizable, i.e. when it is semantically equivalent to an independent set. A partial result to this question is given by

Theorem 4. *(X. Caicedo- [2]) Any theory in $\mathcal{L}_{\omega_1,\omega}$ of cardinality no more than $\aleph_1$ is independently axiomatizable.*

For cardinalities greater than $\aleph_1$, Caicedo obtained partial results for a weaker notion of *countable independence*, which requires that every countable subset of the set of sentences is independent.

Our main result (theorem 26) states that

Theorem 5. *For a countable language $\mathcal{L}$ and for a theory $T \subset \mathcal{L}_{\omega_1,\omega}$, if the number of counterexamples to Vaught's Conjecture contained in T is* small*, then T is independently axiomatizable.*

The meaning of a *small* number of counterexamples is made clear by definition 21. Vaught's conjecture states

Conjecture (Vaught) Every sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$ either has countable many non-isomorphic countable models, or else it has continuum many.

Under the Continuum Hypothesis the conjecture is trivially true and Morley proved that every counterexample to it will necessarily have $\aleph_1$ many non-isomorphic countable models.

Using theorem 5 we then obtain two consequences:

Theorem 6. *Assume Vaught's Conjecture. Then for any countable $\mathcal{L}$ and any theory $T \subset \mathcal{L}_{\omega_1,\omega}$, T is independently axiomatizable.*

Definition 7. *We call a collection of Borel sets $\mathcal{B} = \{B_i | i \in I\}$* independent *if $\bigcap \mathcal{B} \neq \emptyset$ and for every $i \in I$,*

$$\bigcap_{j \neq i} B_j \setminus B_i \neq \emptyset.$$

Two collections $\mathcal{B}, \mathcal{B}'$ are equivalent*, if*

$$\bigcap \mathcal{B} = \bigcap \mathcal{B}'.$$

Theorem 8. *Let $\mathcal{B}$ be a collection of Borel sets. Then there an independent collection of Borel sets $\mathcal{B}'$ with*

$$\bigcap \mathcal{B} = \bigcap \mathcal{B}'.$$

2 Preliminary work

In all that follows we work with sentences in $\mathcal{L}_{\omega_1,\omega}$. If a theory T doesn't have any models, it is axiomatizable by the sentence $\exists x(x \neq x)$, while if it only contains valid sentences, then it is axiomatizable by the empty set. Thus, we can assume that all the theories we work with are consistent and do not contain valid formulas.

Throughout this paper we assume that the language $\mathcal{L}$ we are working with is countable. Then every theory $T \subset \mathcal{L}_{\omega_1,\omega}$ can have size up to the continuum. Under the Continuum Hypothesis and using theorem 4, every theory is independently axiomatizable. So, it suffices to deal with the case that the Continuum Hypothesis fails and we will take this as one of our working assumptions.

Definition 9. *Let ϕ a sentence. We say that the sentences $\{\psi_\alpha | \alpha \in I\}$* partition *$\phi$ if:*

- $\models \phi \leftrightarrow \bigvee_{\alpha \in I} \psi_\alpha$
- *For all α, $\models \psi_\alpha \rightarrow \bigwedge_{\beta \neq \alpha} \neg \psi_\beta$*

What we are heading towards is to prove that under the assumption of a perfect set of countable models, we get a partition into continuum many sentences.

If $\mathcal{M}$ is a countable model and $\vec{a} \in \mathcal{M}$, define the *α-type* of $\vec{a}$ in $\mathcal{M}$ inductively:

$$\begin{aligned}
\phi_0^{\vec{a},\mathcal{M}} &:= \bigwedge\{\psi(\vec{x}) | \psi \text{ is an atomic formula or negation of atomic}, \mathcal{M} \models \psi(\vec{a})\}, \\
\phi_{\alpha+1}^{\vec{a},\mathcal{M}} &:= \phi_\alpha^{\vec{a},\mathcal{M}} \bigwedge\{\exists \vec{y} \phi_\alpha^{\vec{a}\frown\vec{b},\mathcal{M}}(\vec{x},\vec{y}) | \vec{b} \in \mathcal{M}\} \wedge \\
&\quad \bigwedge_n \forall y_0 \dots y_n \bigvee\{\phi_\alpha^{\vec{a}\frown\vec{b},\mathcal{M}}(\vec{x},\vec{y}) | \vec{b} \in \mathcal{M}\}, \\
\phi_\lambda^{\vec{a},\mathcal{M}} &:= \bigwedge_{\alpha<\lambda} \phi_\alpha^{\vec{a},\mathcal{M}}, \text{ for } \lambda \text{ limit.}
\end{aligned}$$

The *α-types of* $\mathcal{M}$ are defined to be all sentences of the form $\phi_\alpha^{\vec{a},\mathcal{M}}$, for any $\vec{a} \in \mathcal{M}$, and if σ is a sentence, the *α-types of σ* are all sentences of the form $\phi_\alpha^{\vec{a},\mathcal{M}}$ with $\mathcal{M} \models \sigma$ and $\vec{a} \in \mathcal{M}$.

If $\mathcal{M}$ is a countable model, then it realizes only countably many types and there is an ordinal $\delta < \omega_1$ such that for all $\vec{a}, \vec{b} \in \mathcal{M}$,

$$\phi_\delta^{\vec{a},\mathcal{M}} \neq \phi_\delta^{\vec{b},\mathcal{M}} \Leftrightarrow \exists \gamma < \omega_1 (\phi_\gamma^{\vec{a},\mathcal{M}} \neq \phi_\gamma^{\vec{b},\mathcal{M}}).$$

The least such ordinal δ we call the Scott height of $\mathcal{M}$ and write $\alpha(\mathcal{M})$. Then $\phi_{\alpha(\mathcal{M})+2}^{\emptyset,\mathcal{M}}$ is called the Scott sentence of $\mathcal{M}$.

Definition 10. *For a $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ and $\alpha < \omega_1$, let*

$$\Psi_\alpha(\phi) := \{\phi_\alpha^{\vec{a},\mathcal{M}} | \vec{a} \in \mathcal{M}, \mathcal{M} \models \phi\},$$

the α-types of ϕ. Define also

$$\Phi_\alpha(\phi) := \{\phi_\alpha^{\emptyset,\mathcal{M}} | \mathcal{M} \models \phi\}.$$

Now, observe that if $\alpha = \gamma + 1$, some γ, then we can identity every $\phi_{\gamma+1}^{\vec{a},\mathcal{M}}$ with the set

$$\{\phi_\gamma^{\vec{a}\frown\vec{b},\mathcal{M}} | \vec{b} \in \mathcal{M}\}.$$

This enables us to consider $\Psi_{\gamma+1}(\phi)$ and $\Phi_{\gamma+1}(\phi)$ as subsets of $X_\alpha(\phi) := 2^{\Psi_\gamma(\phi)}$. In the special case that $\Psi_\gamma(\phi)$ is countable, $X_\alpha(\phi)$ becomes a standard Borel space and we will prove (lemma 13) that in this case $\Psi_{\gamma+1}(\phi)$ and $\Phi_{\gamma+1}(\phi)$ are $\mathbf{\Sigma_1^1}$ subsets.

Similarly, for α limit, we can identify $\phi_\alpha^{\vec{a},\mathcal{M}}$ with the set

$$\{\phi_\gamma^{\vec{a},\mathcal{M}} | \gamma < \alpha\}.$$

Then $\Psi_\alpha(\phi)$ and $\Phi_\alpha(\phi)$ become subsets of $X_\alpha(\phi) := 2^{\bigcup_{\gamma<\alpha} \Psi_\gamma(\phi)}$. Again, in the case that for all $\gamma < \alpha$, $\Psi_\gamma(\phi)$ is countable, $X_\alpha(\phi)$ becomes a standard Borel space and $\Psi_\alpha(\phi)$ and $\Phi_\alpha(\phi)$ are $\mathbf{\Sigma_1^1}$ subsets.

The same can be said for $\phi_0^{\vec{a},\mathcal{M}}$. We can identify it with

$$\{\phi(x) | \phi \text{ atomic}, \mathcal{M} \models \phi(\vec{a})\},$$

in which case $\Psi_0(\phi)$ and $\Phi_0(\phi)$ become subsets of $X_0(\phi) := 2^A$, with A being the set of all atomic, or negation of atomic sentences. Since we assumed that the language we work with is countable, A is countable and $X_0(\phi)$ is a standard Borel space with $\Psi_0(\phi)$ and $\Phi_0(\phi)$ $\mathbf{\Sigma_1^1}$ subsets.

Definition 11. *Let $\mathcal{L}$ be a countable language and let $Mod(\mathcal{L})$ be the set of all countable $\mathcal{L}$-structures with underlying set $\mathbb{N}$. We equip Mod(L) with the topology generated by taking as basic open sets all sets of the form*

$$\{M \in Mod(L) : M \models \varphi(n_1, ...n_m)\}$$

for $\varphi(\vec{x})$ a quantifier free formula and $n_1, ..., n_m \in \mathbb{N}$.

It is easily shown that $Mod(\mathcal{L})$ is a Polish space. For more on this one can consult [4].

Definition 12. *For a sentence σ let $Mod(\sigma)$ be the set of all models in $Mod(\mathcal{L})$ that satisfy σ.*

This becomes a standard Borel space space by the Borel structure it inherits from $Mod(\mathcal{L})$ (cf. [4] too).

Lemma 13. *Let ϕ be a $\mathcal{L}_{\omega_1,\omega}$-sentence, $\alpha < \omega_1$, $\Psi_\alpha(\phi)$, $\Phi_\alpha(\phi)$ and $X_\alpha(\phi)$ as defined above. Assume that for all $\gamma < \alpha$, $\Psi_\gamma(\phi)$ is countable. Then*

1. *the function $Mod(\phi) \times \omega^{<\omega} \to X_\alpha(\phi)$, with*

$$(\mathcal{M}, \vec{a}) \mapsto \phi_\alpha^{\vec{a},\mathcal{M}}$$

   is Borel and

2. *$\Psi_\alpha(\phi)$ and $\Phi_\alpha(\phi)$ are $\mathbf{\Sigma_1^1}$ sets.*

Proof. Recall that under the countability assumption for the Ψ_γ's, $X_\alpha(\phi)$ becomes a standard Borel space with $\Psi_\alpha(\phi)$ and $\Phi_\alpha(\phi)$ seen as subsets of it. Therefore, the statement of the theorem makes sense.

Now, by induction on $\beta \leq \alpha$, it follows easily from the definition that the function $(\mathcal{M}, \vec{a}) \mapsto \phi_\beta^{\vec{a},\mathcal{M}}$ is Borel. In particular, the same is true for $(\mathcal{M}, \vec{a}) \mapsto \phi_\alpha^{\vec{a},\mathcal{M}}$.

Using this function we can write

$$\psi \in \Psi_\alpha \text{ iff } \exists \mathcal{M} \exists \vec{a} \in \mathcal{M}(\mathcal{M} \models \phi \wedge \psi = \phi_\alpha^{\vec{a},\mathcal{M}}),$$

and similarly

$$\psi \in \Phi_\alpha \text{ iff } \exists \mathcal{M}(\mathcal{M} \models \phi \wedge \psi = \phi_\alpha^{\emptyset,\mathcal{M}}).$$

This proves the lemma. □

If $\Psi_\alpha(\phi)$ is as in the above lemma, then by the perfect set theorem for $\mathbf{\Sigma^1_1}$ sets, it is either countable or has size continuum. If it is countable, then we can apply the lemma once more and we can keep doing that until we either run out of countable ordinals, or until we find an uncountable $\Psi_{\alpha'}(\phi)$, some $\alpha' > \alpha$.

Lemma 14. *For all $\alpha < \omega_1$, the set*

$$\{(\mathcal{M}, \mathcal{N}, \vec{a}, \vec{b}) \in Mod(\mathcal{L})^2 \times (\omega^{<\omega})^2 | \phi_\alpha^{\vec{a},\mathcal{M}} = \phi_\alpha^{\vec{b},\mathcal{N}}\}$$

is Borel.

In particular, for $\phi \in \mathcal{L}_{\omega_1,\omega}$ and $\gamma < \omega_1$, the set

$$\{\mathcal{M} \in Mod(\phi) | \alpha(\mathcal{M}) < \gamma\}$$

is also Borel.

Proof. For the first part, by induction on α:

$\alpha = 0$: Then $\phi_0^{\vec{a},\mathcal{M}} = \phi_0^{\vec{b},\mathcal{N}}$ if and only if for every atomic, or negation of atomic, formula ϕ,

$$\mathcal{M} \models \phi(\vec{a}) \Leftrightarrow \mathcal{N} \models \phi(\vec{b}).$$

$\alpha + 1$: Then $\phi_{\alpha+1}^{\vec{a},\mathcal{M}} = \phi_{\alpha+1}^{\vec{b},\mathcal{N}}$ if and only if

$$\forall \vec{c} \in \mathcal{M} \exists \vec{d} \in \mathcal{N} (\phi_\alpha^{\vec{a}\frown\vec{c},\mathcal{M}} = \phi_\alpha^{\vec{b}\frown\vec{d},\mathcal{N}})$$

and

$$\forall \vec{d} \in \mathcal{N} \exists \vec{c} \in \mathcal{M} (\phi_\alpha^{\vec{a}\frown\vec{c},\mathcal{M}} = \phi_\alpha^{\vec{b}\frown\vec{d},\mathcal{N}}).$$

α limit: Then $\phi_\alpha^{\vec{a},\mathcal{M}} = \phi_\alpha^{\vec{b},\mathcal{N}}$ if and only if

$$\forall \beta < \alpha (\phi_\beta^{\vec{a},\mathcal{M}} = \phi_\beta^{\vec{b},\mathcal{N}}).$$

By inductive hypothesis, all these conditions are Borel and therefore our set is Borel.

Now, by the definition of the Scott height, $\alpha(\mathcal{M}) < \gamma$ if and only if

$$\bigvee_{\alpha<\gamma} \forall \vec{a}\, \forall \vec{b}\, (\phi_\alpha^{\vec{a},\mathcal{M}} = \phi_\alpha^{\vec{b},\mathcal{M}} \Rightarrow \phi_{\alpha+1}^{\vec{a},\mathcal{M}} = \phi_{\alpha+1}^{\vec{b},\mathcal{M}}).$$

By the first part, this condition is Borel. □

Lemma 15. *If a $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ has continuum many non-isomorphic countable models, then there are countable ordinals $\alpha < \beta$, with $X_\alpha(\phi)$ a standard Borel space, a perfect set P and continuous functions $t : P \to X_\alpha(\phi)$, $M : P \to Mod(\phi)$ such that:*

- *for all $x \neq y \in P$, $t(x), t(y)$ are distinct types in $X_\alpha(\phi)$,*
- *for all $x \in P$, $M(x)$ is a countable model of ϕ that realizes $t(x)$ and has Scott height $< \beta$.*

Moreover, by restricting P to a perfect subset P_1, we can assume that for $x \neq y \in P_1$, $M(x) \not\models t(y)$.

Proof. Let $\alpha < \omega_1$ be the least ordinal with $\Phi_\alpha(\phi)$ uncountable. Then $\Phi_\gamma(\phi)$, $\gamma < \alpha$, are all countable and applying lemma 13, we conclude that $X_\alpha(\phi)$ is a standard Borel space and $\Phi_\alpha(\phi)$ is $\mathbf{\Sigma^1_1}$. Consider an ordinal $\beta > \alpha$ large enough so that the set

$$\{\phi_\alpha^{\emptyset,\mathcal{M}} \in \Phi_\alpha(\phi) | \alpha(\mathcal{M}) < \beta\}$$

is still uncountable. By lemma 14 this is again $\mathbf{\Sigma^1_1}$. Consequently, it embeds a perfect set.

So, let P a perfect set with

$$t : P \to \{\phi_\alpha^{\emptyset,\mathcal{M}} \in \Phi_\alpha(\phi) | \alpha(\mathcal{M}) < \beta\} \subset X_\alpha(\phi)$$

a continuous embedding. Then every $t(x)$, $x \in P$, has the form $\phi_\alpha^{\emptyset,\mathcal{M}}$, for some $\mathcal{M}$ with $\alpha(\mathcal{M}) < \beta$.

Consider the set

$$\{(x, \mathcal{M}) \in P \times Mod(\phi) | \mathcal{M} \models t(x), \text{ Scott height}(\mathcal{M}) < \beta\}.$$

This is not empty and by lemma 14 and since t is continuous, it is Borel. By Jankov- von Neumann Uniformization theorem (cf. [1]), we get a function $x \mapsto M(x)$ that is Baire measurable and for all $x \in P$, $(x, M(x))$ is in the above set. Restricting the domain to a comeager set $C \subset P$ we can further assume that $x \mapsto M(x)$ is continuous on C.

Let

$$R_0 := \{(x, y) \in C^2 | M(x) \not\models t(y)\}.$$

Since t is 1-1 and $M(x)$ can satisfy only countably many α-types, R_0 is comeager in C^2. By theorem 19.1 of [1], we can find a Cantor set $C_1 \subset C$ such that

$$(C_1)^2 \subset R_0,$$

or that for all $x, y \in C_1$,

$$x \neq y \Rightarrow (x, y) \in R_0 \Rightarrow M(x) \not\models t(y),$$

which proves the lemma. □

Observe here that for $x \neq y \in P_1$, $M(x) \models t(x)$, while $M(y) \not\models t(x)$, which implies that $M(x) \ncong M(y)$.

Lemma 16. *The set $A_0 := \{\mathcal{M} | \exists x \in P_1 (\mathcal{M} \cong M(x))\}$ is Borel.*

Proof. We need first that the set $A_1 := \{(x, \mathcal{M}) | \mathcal{M} \cong M(x), x \in P_1\}$ is Borel. Since for all $x \in P_1$ the Scott height of $M(x)$ is $< \beta$,

$$\begin{aligned} \mathcal{M} \cong M(x) \quad & iff \quad \mathcal{M} \models \phi_{\beta+1}^{\emptyset, M(x)} \\ & iff \quad \phi_{\beta+1}^{\emptyset,\mathcal{M}} = \phi_{\beta+1}^{\emptyset, M(x)}. \end{aligned}$$

This last condition is Borel by lemma 14.

By the observation that for $x \neq y \in P_1$, $M(x) \ncong M(y)$, we can also conclude that if $(x_1, \mathcal{M}), (x_2, \mathcal{M})$ are both in P_1, then $x_1 = x_2$. By the Lusin-Novikov theorem, the projection of A_1 (on the second component) is also Borel and this is exactly what we have to prove. □

Corollary 17. *There is a sentence $\phi^+ \in \mathcal{L}_{\omega_1,\omega}$ such that for every countable model $\mathcal{M}$,*

$$\mathcal{M} \models \phi^+ \Leftrightarrow \mathcal{M} \in A_0.$$

Proof. A_0 is obviously invariant under isomorphisms and by the previous lemma is Borel. Therefore, there exists a $\mathcal{L}_{\omega_1,\omega}$ sentence ϕ^+ as in the statement. □

Lemma 18. *If $\mathcal{N}$ is a model of ϕ^+, countable or uncountable, and it satisfies one of the α-types $\{t(x)|x \in P_1\}$, then it actually satisfies the Scott sentence $s(x)$ of $M(x)$.*

Proof. Recall here that a countable model can satisfy only one of the α-types $\{t(x)|x \in P_1\}$.

If $\mathcal{N}$ is countable and satisfies ϕ^+, then it belongs to A_0, i.e. it is isomorphic to one of the $M(x)$, $x \in P_1$. If $\mathcal{N} \models t(x)$, then $M(x) \cong \mathcal{N}$ and obviously $\mathcal{N} \models s(x)$.

Therefore, assume that $\mathcal{N}$ is uncountable with $\mathcal{N} \models t(x)$, some $x \in P_1$. Let $s(x)$ be the Scott sentence of $M(x)$ and $\mathcal{F}$ the fragment generated by $\phi^+, t(x)$ and $s(x)$. Let $\mathcal{N}_0$ be a countable model with

$$\mathcal{N}_0 \prec_{\mathcal{F}} \mathcal{N}.$$

Then $\mathcal{N}_0 \models \phi^+$ and $\mathcal{N}_0 \models t(x)$. As in the countable case, $\mathcal{N}_0 \models s(x)$, which implies that $\mathcal{N} \models s(x)$. □

Using all these lemmas we are ready to prove

Theorem 19. *If ϕ has $2^{\aleph_0}$ many non-isomorphic countable models, then there exists a partition of ϕ into continuum many sentences.*

Proof. Assume that $P_1, \alpha, x \mapsto t(x), x \mapsto M(x)$ and ϕ^+ are as above.

Claim 1. *It suffices to find a $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ^* that expresses the fact that our model satisfies one of the α-types $\{t(x)|x \in P_1\}$.*

Proof. (of claim) First we need that every model of ϕ^+ is also a model of ϕ^*. Arguing as before let $\mathcal{N} \models \phi^+$, $\mathcal{F}$ be the fragment generated by both ϕ^+ and ϕ^*, and $\mathcal{N}_0 \prec_{\mathcal{F}} \mathcal{N}$ a countable model. Then, there exists $x \in P_1$ with $\mathcal{N}_0 \cong M(x)$ and $\mathcal{N}_0 \models t(x)$. By definition, $\mathcal{N}_0 \models \phi^*$ which also implies that $\mathcal{N} \models \phi^*$.

Combining this with the previous lemma, we conclude that every countable or uncountable model of ϕ^+ will satisfy one of the Scott sentences $\{s(x)|x \in P_1\}$. Therefore,

$$\{\phi \wedge \neg\phi^+\} \cup \{s(x)|x \in P_1,\ s(x) \text{ is the Scott sentence of } M(x)\}$$

gives a partition of ϕ into continuum many sentences. □

Towards constructing ϕ^*, let $S := \bigcup_{\gamma<\alpha} \Psi_\gamma(\phi)$. By assumption on α, S is countable and for all $x \in P_1$, $t(x) \in X_\alpha(\phi) \subset 2^S$.

For all $u \in 2^{<\omega}$ we can construct S_u finite subsets of S such that

1. for every u, $S_{u\frown 0}$ is always incompatible with $S_{u\frown 1}$,

2. $S_u \subset S_w$ when $u \subset w$,
3. for every $\hat{u} \in 2^\omega$, $\bigcup_{n\in\mathbb{N}} S_{\hat{u}\restriction n}$ is an element of $\{t(x)|x \in P_1\}$ and every $t(x)$ in this set can be written as $\bigcup_{n\in\mathbb{N}} S_{\hat{u}\restriction n}$, for some $\hat{u} \in 2^\omega$.

Consider the sentence:

$$\phi^* := \exists a \bigwedge_{n\in\omega} \bigvee_{u\in 2^n} \bigwedge_{\psi\in S_u} \psi(a).$$

It is obvious that every model of ϕ^* will satisfy one of the α-types $t(x)$, $x \in P_1$. □

3 Main Result

We work as before with a countable $\mathcal{L}$. Throughout this section we will not distinguish between a model $\mathcal{M}$ and its isomorphism class $[\mathcal{M}]_{S_\infty}$. So, when we say that a sentence has countably many countable models, we actually mean countably many non-isomorphic countable models.

Definition 20. *For a theory $T = \{\phi_\alpha|\alpha < 2^{\aleph_0}\}$ define*

$$\begin{aligned} T_0 &:= \{\phi \in T|\neg\phi \text{ has countable many countable models}\}, \\ T_1 &:= \{\phi \in T|\neg\phi \text{ has } \aleph_1 \text{ many countable models}\}, \\ T_2 &:= \{\phi \in T|\neg\phi \text{ has } 2^{\aleph_0} \text{ many countable models}\}, \end{aligned}$$

and

$$\begin{aligned} X_0(T) &:= \{\mathcal{M}|\mathcal{M} \models \neg\phi, \text{ some } \phi \in T_0,\ \mathcal{M} \text{ countable}\} \\ X_1(T) &:= \{\mathcal{M}|\mathcal{M} \models \neg\phi, \text{ some } \phi \in T_1,\ \mathcal{M} \text{ countable}\} \\ X_2(T) &:= \{\mathcal{M}|\mathcal{M} \models \neg\phi, \text{ some } \phi \in T_2,\ \mathcal{M} \text{ countable}\} \\ X(T) &:= X_0(T) \cup X_1(T) \cup X_2(T). \end{aligned}$$

Note that the sets T_0, T_1 and T_2 are disjoint, while the sets $X_0(T), X_1(T)$ and $X_2(T)$ may not be disjoint. Also, all sentences in T_1 provide counterexamples to Vaught's Conjecture.

Definition 21. *In case that $|X(T)| \geq |T_1|$, we say that T_1 is* small.

Smallness assumption for T_1 will be crucial for our main result (theorem 26). Now, if $|X(T)| = \aleph_0$, then $T_1 = T_2 = \emptyset$ and if $\{\mathcal{M}_n|n \in \omega\}$ enumerate the models in X_0 and $\{\phi_n|n \in \omega\}$ enumerate their Scott sentences, then it is easily seen that

$$T \Leftrightarrow \bigwedge \neg\phi_n.$$

So, we can assume that $|X(T)|$ is uncountable.

We will split the proof in various cases given by corresponding lemmas.

Lemma 22. *If $X_2(T) \neq \emptyset$, then T is independently axiomatizable.*

Proof. In this case there is a sentence, say ϕ_0, such that $\neg\phi_0$ has continuum many non-isomorphic countable models.

By theorem 19 we know that there are sentences $\{\psi_\alpha | 0 < \alpha < 2^{\aleph_0}\}$ that partition $\neg\phi_0$. Define a new theory[1]

$T' = \{\overline{\phi_\alpha} | 0 < \alpha < 2^{\aleph_0}\}$ by

$$\overline{\phi_\alpha} : \quad (\neg\phi_0 \Rightarrow \neg\psi_\alpha) \wedge (\phi_0 \Rightarrow \phi_\alpha).$$

Claim 2. *T and T' are semantically equivalent.*

If we assume T, then for every $\overline{\phi_\alpha}$ the second part of its conjunction is trivially true, while the first part is true since its antecedent is false.

Now, assume T'. If ϕ_0 holds, then for every $\overline{\phi_\alpha}$ the second part of its conjunction implies ϕ_α. Therefore, T holds. On the other hand, if ϕ_0 fails, then the conjunction of all the $\overline{\phi_\alpha}$'s implies

$$\bigwedge_\alpha \neg\psi_\alpha.$$

But by the way the ψ_α's were defined,

$$\models \neg\phi_0 \Leftrightarrow \bigvee_\alpha \psi_\alpha.$$

Combining these two we get ϕ_0. Contradiction. Therefore, ϕ_0 can not fail.

Claim 3. *T' is independent.*

Let $\alpha < 2^{\aleph_0}$ and assume that there is a model $\mathcal{M}_\alpha$ with $\mathcal{M}_\alpha \models \psi_\alpha$. By the assumption that the ψ_α's partition $\neg\phi_0$, we get that $\mathcal{M}_\alpha \models \neg\phi_0$ and for all other $\beta \neq \alpha$, $\mathcal{M}_\alpha \models \neg\psi_\beta$. Therefore,

$$\mathcal{M}_\alpha \models \bigwedge_{\beta\neq\alpha} \overline{\phi_\beta} \wedge \neg\overline{\phi_\alpha}.$$

This means that $T' \setminus \{\overline{\phi_\alpha}\} \not\models \overline{\phi_\alpha}$, or that T' is independent.

□

Lemma 23. *If $X_2(T) = \emptyset$ and $|X_0(T) \setminus X_1(T)| = |X(T)| \geq |T_1|$, then T is independently axiomatizable.*

Proof. Before we start we need a lemma that essentially is due to Reznikoff (cf. [3]) and also appears in [2]. We include the proof for completeness.

Lemma 24. *Let C, D be disjoint sets of sentences with $|D| \leq |C|$. If every $\phi \in C$ is not implied by other sentences of $C \cup D$, then $C \cup D$ is equivalent to an independent theory.*

Proof. Let $f : D \to C$ be a 1-1 function. Then the set

$$(C \setminus f(D)) \cup \{\phi \wedge f(\phi) | \phi \in D\}$$

is independent and semantically equivalent to $C \cup D$. □

[1] Note here that both ψ_α and $\overline{\phi_\alpha}$ are defined for $\alpha > 0$.

Now, assume that $|X(T)| = \kappa \geq \omega_1$.

By the previous lemma it suffices to find a theory T_0' such that

- $T_0' \cup T_1$ is equivalent to $T_0 \cup T_1$,
- $|T_0'| = \kappa \geq |T_1|$ and
- Every sentence in T_0' is not implied by other sentences in $T_0' \cup T_1$.

Let $T_0 = \{\phi_\alpha | \alpha < 2^{\aleph_0}\}$ and for every α, let $\{\mathcal{M}_n^{(\alpha)} | n \in \mathbb{N}\}$ and $\{\phi_n^{(\alpha)} | n \in \mathbb{N}\}$ be an enumeration of the (countably many) countable models of $\neg\phi_\alpha$ and their Scott sentences respectively. Define

$$\overline{\phi_\alpha} = \bigwedge \{\neg\phi_n^{(\alpha)} | \mathcal{M}_n^{(\alpha)} \notin X_1(T), \phi_n^{(\alpha)} \neq \phi_m^{(\beta)}, \beta < \alpha, m \in \mathbb{N}\},$$

i.e. we get the conjunction of all the Scott sentences that neither did they appear at a previous step nor their countable model is in $X_1(T)$. If the conjunction is empty we ignore it. By assumption $|X_0(T) \setminus X_1(T)| = |X(T)| = \kappa$ and there have to be κ many $\overline{\phi_\alpha}$'s that are not empty. Let $T_0' = \{\overline{\phi_\alpha} | \alpha < \kappa\}$.

Claim 4. *$T_0' \cup T_1$ is equivalent to $T_0 \cup T_1$.*

First observe that

$$\neg\phi_\alpha \Leftrightarrow \bigvee_n \phi_n^{(\alpha)},$$

or that

$$\phi_\alpha \Leftrightarrow \bigwedge_n \neg\phi_n^{(\alpha)}.$$

Thus, $\phi_\alpha \Rightarrow \overline{\phi_\alpha}$, or that

$$T_0 \cup T_1 \models T_0' \cup T_1.$$

Conversely, let $\mathcal{M} \models T_0' \cup T_1$. We need to prove that $\mathcal{M} \models T_0$, which is equivalent to

$$\mathcal{M} \models \phi_\alpha, \text{ for all } \alpha,$$

or that

$$\mathcal{M} \models \bigwedge_n \neg\phi_n^{(\alpha)}, \text{ for all } \alpha,$$

or

$$\mathcal{M} \models \neg\phi_n^{(\alpha)}, \text{ for all } \alpha \text{ and } n.$$

Hence, assume that $\mathcal{M} \models \phi_n^{(\alpha)}$, for some α and n. Since $\mathcal{M} \models T_0'$, the only case that this can happen is if $\mathcal{M}_n^{(\alpha)} \in X_1(T)$. If $\mathcal{M}$ is not countable, we can pass to a countable elementary submodel (over an appropriate fragment), say $\mathcal{M}_0 \prec \mathcal{M}$. Then $\mathcal{M}_0 \cong \mathcal{M}_n^{(\alpha)}$ and, therefore, there is $\phi \in T_1$ with $\mathcal{M}_0 \models \neg\phi$. If the fragment was chosen to include ϕ, we also get that $\mathcal{M} \models \neg\phi$, contradicting the fact that $\mathcal{M} \models T_1$.

Claim 5. *Every sentence in T_0' is not implied by other sentences in $T_0' \cup T_1$.*

Fix α and assume that $\overline{\phi_\alpha}$ is not empty, with, say $\neg\phi_n^{(\alpha)}$, being one sentence in the conjunction. Since $\mathcal{M}_n^{(\alpha)} \models \phi_n^{(\alpha)}$, it cannot satisfy any other Scott sentence and since $\neg\phi_n^{(\alpha)}$ doesn't appear in any other $\overline{\phi_\beta}$, we conclude that

$$\mathcal{M}_n^{(\alpha)} \models \neg\overline{\phi_\alpha} \wedge \bigwedge_{\beta \neq \alpha} \overline{\phi_\beta}.$$

So, $\mathcal{M}_n^{(\alpha)} \models T_0' \setminus \{\overline{\phi_\alpha}\}$. But also, for every $\phi \in T_1$, $\mathcal{M}_n^{(\alpha)} \models \phi$, because otherwise it would be $\mathcal{M}_n^{(\alpha)} \in X_1$ and this would prevent $\neg\phi_n^{(\alpha)}$ from being in the conjunction of $\overline{\phi_\alpha}$. Contradiction.

Putting everything together we get that

$$\mathcal{M}_n^{(\alpha)} \models T_0' \cup T_1 \setminus \{\overline{\phi_\alpha}\},$$

witnessing that $T_0' \cup T_1 \setminus \{\overline{\phi_\alpha}\} \not\models \overline{\phi_\alpha}$.

This establishes the claim and finishes the proof. □

Lemma 25. *If $X_2 = \emptyset$ and $|X(T)| \geq |T_1|$, then T is independently axiomatizable.*

Proof. If $|X(T)| = \kappa$ and $|X_1(T)| < \kappa$, then the assumptions of lemma 23 are satisfied and T is independently axiomatizable. So, assume that $X_1(T) = \{\mathcal{M}_\alpha | \alpha < \kappa\}$ with $\kappa \geq \omega_1$ and $T_1 = \{\psi_\alpha | \alpha < \lambda\}$ with $\lambda \leq \kappa$.

We will find another theory $T^\star$, equivalent to T and for which

$$|X_0(T^\star) \setminus X_1(T^\star)| = |X(T^\star)| = |X(T)| \geq |T_1| = |T_1^\star|.$$

Again, by lemma 23 we are done.

We know that the only case that a sentence ϕ can be in T_1 is if for all countable α, both $\Phi_\alpha(\neg\phi)$ and $\Psi_\alpha(\neg\phi)$ are countable. For every $\alpha < \omega_1$ define new sets $C_\alpha(\neg\phi)$ and $S_\alpha(\neg\phi)$:

$\phi_\alpha^{\emptyset,\mathcal{M}} \in C_\alpha(\neg\phi)$ if and only if $\phi_\alpha^{\emptyset,\mathcal{M}} \in \Phi_\alpha(\neg\phi)$ and there are only countably many countable models of ϕ that satisfy $\phi_\alpha^{\emptyset,\mathcal{M}}$, and

$\sigma \in S_\alpha(\neg\phi)$ if and only if there exists a countable model $\mathcal{M}$ that satisfies some $\phi_\alpha^{\emptyset,\mathcal{M}} \in C_\alpha(\neg\phi)$ and σ is its Scott sentence.

Both $C_\alpha(\neg\phi)$ and $S_\alpha(\neg\phi)$ are countable for all α, since $\Phi_\alpha(\neg\phi)$ is countable.

We will distinguish three cases:

Case I: $\kappa > \omega_1$ and $cf(\kappa) \neq \omega_1$.

Since

$$\kappa = |X_1(T)| \leq |T_1|\aleph_1 = \lambda \cdot \aleph_1$$

and

$$\kappa > \omega_1,$$

it must be that $\kappa = \lambda$. Since $cf(\kappa) \neq \omega_1$, there exists an ordinal $\gamma < \omega_1$ and κ non-isomorphic countable models in X_1 of Scott height less than γ.

Define inductively a new theory, considering the sentence $\neg\psi_\alpha$ at stage $\alpha < \lambda$. Choose β larger than γ and replace ψ_α by

$$\psi_\alpha^{(0)} := \bigwedge\{\neg\sigma | \sigma \in S_\beta(\neg\psi_\alpha)\}$$

and

$$\psi_\alpha^{(1)} := \psi_\alpha \vee \bigvee\{\sigma | \sigma \in S_\beta(\neg\psi_\alpha)\}.$$

It is not hard to see that ψ_α is equivalent to the conjunction of $\psi_\alpha^{(0)}$ and $\psi_\alpha^{(1)}$. Also, observe that $\neg\psi_\alpha^{(0)}$ has countably many countable models and all the countable models of $\neg\psi_\alpha$ of Scott height less than γ satisfy it.

Repeating this for λ many steps we will get eventually a theory $T^\star$ such that $X_0(T^\star)$ will contain all countable models that are in $X_1(T)$ that have Scott height $< \gamma$. By the assumption on γ,

$$|X_0(T^\star) \setminus X_1(T^\star)| = \kappa.$$

Case II: $\kappa > \omega_1$ and $cf(\kappa) = \omega_1$.

As before $\kappa = \lambda$, but the difference now is that we may not have an ordinal γ as before. Instead, assume that there are cardinals $\{\mu_i | i < \omega_1\}$ and countable ordinals $\{\alpha_i | i < \omega_1\}$ such that

- for all $i < j$, $\omega_1 < \mu_i < \mu_j$,
- $\sup_i \mu_i = \kappa$,
- for $i = 0$, $\alpha_0 = 0$,
- for all $i < j$, $\alpha_i < \alpha_j$, and
- for j limit ordinal, $\sup_{i<j} \alpha_i = \alpha_j$, and
- for all $i < \omega_1$, the number of countable models in $X_1(T)$ that have Scott height α with $\alpha_i \le \alpha < \alpha_{i+1}$ is equal to μ_i.

This also yields a partition $T_1 = \bigcup_{i<\omega_1} T_1^{(i)}$ such that for all i

- for all $\psi \in T_1^{(i)}$, $\neg\psi$ has a countable model of Scott height α, $\alpha_i \le \alpha < \alpha_{i+1}$ and
- $|\{\mathcal{M} | \mathcal{M} \models \neg\psi$, some $\psi \in T_1^{(i)}$, $\mathcal{M}$ countable and $\alpha_i \le \alpha(\mathcal{M}) < \alpha_{i+1}\}| = \mu_i$.

As before we define a new theory inductively: At stage α, if $\psi_\alpha \in T_1^{(i)}$, choose $\beta \ge \alpha_{i+1}$ and replace ψ_α by

$$\psi_\alpha^{(0)} := \bigwedge\{\neg\sigma | \sigma \in S_\beta(\neg\psi_\alpha)\}$$

and

$$\psi_\alpha^{(1)} := \psi_\alpha \vee \bigvee\{\sigma | \sigma \in S_\beta(\neg\psi_\alpha)\}.$$

It is not hard to see that ψ_α is equivalent to the conjunction of $\psi_\alpha^{(0)}$ and $\psi_\alpha^{(1)}$. Also, $\neg\psi_\alpha^{(0)}$ has countably many countable models, while $\neg\psi_\alpha^{(1)}$ has $\aleph_1$ many countable models, and all the countable models of $\neg\psi_\alpha$ of Scott height $< \alpha_{i+1} \le \beta$ satisfy $\neg\psi_\alpha^{(0)}$.

Eventually, after λ many steps we will get a theory $T^\star$ such that $X_0(T^\star) \setminus X_1(T^\star)$ contains at least μ_i many countable models $\mathcal{M} \in X_1(T)$ that have Scott height $\alpha_i \le \alpha(\mathcal{M}) < \alpha_{i+1}$. By the assumptions on the μ_i's,

$$|X_0(T^\star) \setminus X_1(T^\star)| = \kappa.$$

Case III: $\kappa \le \omega_1$.

Then $\lambda \le \omega_1$ and we can use Caicedo's theorem (in [2]) that every set with $\le \omega_1$ sentences in $\mathcal{L}_{\omega_1,\omega}$ is independently axiomatizable. □

Theorem 26. *If $|X(T)| \ge |T_1|$, then T is independently axiomatizable.*

Proof. If $X_2 \neq \emptyset$, then use lemma 22. If it is empty, then use the previous lemma. □

Corollary 27. *If the Vaught Conjecture holds, then every $T \subset \mathcal{L}_{\omega_1,\omega}$ is independently axiomatizable.*

Proof. The Vaught Conjecture gives us that $T_1 = \emptyset$. Then use the previous theorem. $\square$

Corollary 28. *If $|X(T)| = 2^{\aleph_0}$, then T is independently axiomatizable.*

Proof. Then $|X(T)| \geq |T_1|$ and we can again apply theorem 26. $\square$

4 Reformulations and open questions

In this section we reformulate the previous theorems as statements about Borel sets and give some open problems.

Recall that a collection of Borel sets $\mathcal{B} = \{B_i | i \in I\}$ is independent if $\bigcap \mathcal{B} \neq \emptyset$ and for every $i \in I$, $\bigcap_{j \neq i} B_j \setminus B_i \neq \emptyset$, and that two collections $\mathcal{B}, \mathcal{B}'$ are equivalent if $\bigcap \mathcal{B} = \bigcap \mathcal{B}'$.

Theorem 29. *Every collection of Borel sets $\mathcal{B} = \{B_i | i \in 2^{\aleph_0}\}$ with $\bigcap \mathcal{B} \neq \emptyset$ admits an equivalent independent collection.*

Proof. The proof closely resembles the proofs of lemma 23 and lemma 22. We have two cases:

Case I: There is an $i_0 \in I$, such that $\complement B_{i_0}$, the complement of B_{i_0}, is uncountable.

Then we can partition $\complement B_{i_0}$ into continuum many sets $\bigcup_{x \in \complement B_{i_0}} \{x\}$. Call these sets $\{C_j | j \neq i_0, j < 2^{\aleph_0}\}$. Define now a new collection of Borel sets $\mathcal{B}' = \{\mathcal{B}'_j | j \neq i_0, j < 2^{\aleph_0}\}$ by

$$\mathcal{B}'_j := (B_{i_0} \vee \complement C_j) \wedge (\complement B_{i_0} \vee B_j).$$

Claim 6. *$\mathcal{B}$ and $\mathcal{B}'$ are equivalent.*

Let $x \in \bigcap \mathcal{B}$, or that $x \in B_i$, all $i < 2^{\aleph_0}$. It follows from the definition of B'_j that $x \in B'_j$, all j, or that $x \in \bigcap \mathcal{B}'$.

So, assume that $x \in \bigcap \mathcal{B}'$. If $x \in B_{i_0}$, then the second conjunct will imply that $x \in B_j$, for all $j \neq i_0$. Therefore, $x \in \bigcap \mathcal{B}$. If on the other hand, $x \notin B_{i_0}$, then the first conjunct will imply that $x \in \complement C_j$, all $j \neq i_0, j < 2^{\mathbb{N}}\}$. But the C_j's partition $\complement B_{i_0}$, which means that $x \in B_{i_0}$. Contradiction. So, this case can't happen.

Claim 7. *$\mathcal{B}'$ is independent.*

Let $x \in C_j$. By the properties of the C_j's, we get that $x \in \complement B_{i_0}$ and $x \notin C_{j'}$, for $j' \neq j$. Therefore, $x \notin B'_j$, while $x \in B'_{j'}$, $j' \neq j$, which implies that

$$\bigcap_{j' \neq j} B'_{j'} \setminus B'_j \neq \emptyset.$$

Case II: For all $i \in I$, $\complement B_i$, the complement of B_i, is countable.

Construct a new collection $\mathcal{B}' = \{B'_j | j < 2^{\aleph_0}\}$ with

$$B'_j = B_j \bigcup_{i<j} \complement B_i.$$

If the set is equal to the whole space, we ignore it and proceed to the next one. Observe here that the the complement of B'_j is a subset of the complement of B_j, which is countable by assumption. Therefore, it is Borel.

Claim 8. *$\mathcal{B}$ and $\mathcal{B}'$ are equivalent.*

It is immediate that $B'_j \supset B_j$, which means that $\bigcap_j B'_j \supset \bigcap_j B_j$. So, let $x \in \bigcap_j B'_j$. By induction on j we can prove that $x \in B_j$. Assume that $x \in \bigcap_{i<j} B_i$. Then $x \notin \bigcup_{i<j} \complement B_i$. Since, $x \in B'_j$, this implies that $x \in B_j$. Consequently, $x \in \bigcap_j B_j$.

Claim 9. *$\mathcal{B}'$ is independent.*

Fix $j < 2^{\aleph_0}$ and assume that B'_j is not equal to the whole space. Say $y \in \complement B'_j$ witnesses this. Then, $y \in \complement B_j$ and by definition $y \in B'_i$, for all $i > j$.

Similarly, $y \notin B'_j$ implies that $x \notin \bigcup_{i<j} \complement B_i$, which means that $x \in \bigcap_{i<j} B_i$. Then, $x \in B'_i$, for all $i < j$, and overall, $y \in \bigcup_{i \neq j} B'_i \setminus B'_j$.

In either case, we constructed an independent collection of Borel sets $\mathcal{B}'$ with is equivalent to $\mathcal{B}$. □

It would be interesting if we could derive theorem 26 from theorem 29. This would eliminate the extra assumptions of theorem 26.

Definition 30. *Let $T \models_g \phi$ mean that in all generic extensions every model of T is also a model of ϕ.*

This is a stronger notion than $T \models \phi$ and is related to $T \vdash_{\mathcal{L}_{\omega_1,\omega}} \phi$, but we will not define $\vdash_{\mathcal{L}_{\omega_1,\omega}}$ here. We can prove

Theorem 31. *If $T \models_g \phi$, then there are countably many sentences $\phi_0, \phi_1, \ldots \in T$ such that*

$$\bigwedge_n \phi_n \models_g \phi.$$

We now ask whether we can replace $\models$ by $\models_g$ in theorem 26. The problem is that T and T' may not be semantically equivalent in a generic extension. This is an open question we did not consider.

We can also reformulate this problem using the language of Boolean Algebras. We know that the $\mathcal{L}_{\omega_1,\omega}$- sentences form a σ-complete Boolean Algebra with $\phi \leq \psi$ if and only if $\phi \to \psi$. Using theorem 31 we can prove that the σ-filter generated by a theory T is equal to

$$T' = \{\psi | T \models_g \psi\}.$$

Definition 32. *A set A of sentences is called σ-*filter independent*, if for all ϕ, ϕ is not in the σ-filter generated by $A \setminus \{\phi\}$.*

The problem is given a set of sentences A to find another set A' such that

- A and A' generate the same σ-filter and
- A' is σ-filter independent.

We can also extend the question to finding conditions under which a Boolean Algebra satisfies the above statement. As far as we know this problem is open.

Another extension would be to prove that any $\mathcal{L}_{\omega_1,\omega}$ theory is independently axiomatizable, without assuming countability of the language. Our techniques here rely heavily on this assumption.

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[5] http://www.math.wisc.edu/ miller/res/problem.pdf

This webpage contains a list of intersting problems in Set Theory and Model Theory.