Free continuous actions on zero-dimensional spaces

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Abstract

We show that every countably infinite group admits a free, continuous action on the Cantor set having an invariant probability measure. We also show that every countably infinite group admits a free, continuous action on a non-homogeneous compact metric space and the action is *minimal* (that is to say, every orbit is dense). In answer to a question posed by Giordano, Putnam and Skau, we establish that there is a continuous, minimal action of a countably infinite group on the Cantor set such that no *free* continuous action of any group gives rise to the same equivalence relation.

0 Introduction

In this paper we consider various dynamical problems which are well understood in the Borel and measure-theoretic contexts, but less explored in the case of continuous actions on zero-dimensional compact metric spaces.

Recall that every non-empty zero-dimensional compact metric space either has an isolated point or is isomorphic to Cantor space. Since we will be considering continuous actions on such spaces, along with the natural assumptions that there is more than one orbit and *at least* one of the orbits is dense, we may already dismiss the possibility of any isolated points. Thus from this point of view the topological dynamics of zero-dimensional spaces reduces to the dynamics of continuous actions on Cantor space.

Note moreover that the decision to work with zero-dimensional spaces minimizes the topological contribution of the space itself. The homotopy and homology is necessarily trivial. Cantor space is homogeneous – which is to say, the full homeomorphism group acts transitively. There is a canonical basis for the space, any two members of which are homeomorphic. In this respects it resembles the situation of standard Borel spaces, where any two uncountable Borel sets are Borel isomorphic and the Borel structure itself is extremely malleable.

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Thus, since the topology is in itself trivial, all the complexity comes from the action of the group itself.

Throughout this paper the term *countable group* will mean a countably infinite group.

Our first result (Section 1, Theorem 1.1) is to show that any countable group admits actions in the given class, and that one may further find an invariant measure.

Theorem 0.1 *Any countable group G admits a continuous and free action on Cantor space with an invariant Borel probability measure.*

Appealing to Giordano and de la Harpe at 3.1 this gives a new characterization of amenability.

In answer to a question from Giordano, Putnam, and Skau, we prove (Section 4, Corollary 4.2) that there are equivalence relations arising from actions in our class which cannot be realized as arising from *free actions*.

Theorem 0.2 *There is a continuous, minimal action of a countable group Γ on Cantor space which cannot be induced by a continuous free action of any countable group.*

This can be viewed as a topological version of the Adams counterexample, to the effect that there are Borel actions of countable groups on standard Borel spaces (with infinite orbits) with orbit equivalence relations which are not induced by a free Borel action of any group. This could equally be viewed as an analogue of Furman's much deeper result that there are measurable actions of countable groups (with infinite orbits) whose orbit equivalence relation cannot be realized by an (essentially) free measurable action of any group.

We also obtain one result by starting with a free continuous action of a countable group on Cantor space and lifting to a kind of extension of the original action. Thus while the next result (Section 2, Theorem 2.1) does not literally mention actions on Cantor space, the argument necessarily makes use of such ideas.

Theorem 0.3 *Any countable group admits a free, continuous, minimal action on a non-homogeneous compact, metric space.*

1 Free actions and invariant measures on the Cantor set

Throughout this section G is a countable discrete group with identity denoted by e. On the notational side we will, whenever convenient, write (X, G, α) meaning the continuous action of G on the topological space X by the homomorphism $\alpha : G \longrightarrow Homeo(X)$. Throughout the text, when referring to an *action* we shall mean a continuous action as defined above.

The following will be our main goal in this section.

Theorem 1.1 *Let G be a countable, discrete group and let X be the Cantor set. Then there exists a free, continuous action (X, G, α) having a G-invariant probability measure.*

Remark 1.2 The existence of a free, continuous action on a compact metric space was previously established by R. Ellis [2] by a construction involving the Stone-Čech compactification. (Although the Stone-Čech compactification yields a non-separable space, one obtains a compact metric space by a standard modification.) Later Veech [8] and Pestov [7] showed similar results for locally compact groups as well as for other topological groups. However, all of these constructions give no information about invariant measures. Our construction yields invariant measures, and is an elaboration of an idea suggested to us by S. Mozes. We want to express our gratitude to him.

Let a countable discrete group G be given. We will divide the discussion into two parts. First we construct a free action of G on the Cantor set. Secondly, we show that our construction naturally yields an invariant probability measure.

Fix some $g \in G$, and let $\langle g \rangle$ denote the cyclic subgroup of G generated by g. Choose a sequence $\{\gamma_i\}_{i \in I_g}$ in G, where the index set I_g is either $\{0, 1, 2, ..., n\}$, or $\{0, 1, 2, ...\}$ such that

$$G = \bigsqcup_{i \in I_g} \gamma_i \langle g \rangle.$$

That is, we write G as a disjoint union of left cosets of the cyclic subgroup $\langle g \rangle$. We set $\gamma_0 = e$ independent of the choice of g. For every aperiodic $g \in G$ (i.e. $\langle g \rangle$ is infinite) fix a Cantor minimal system (X_g, T_g), and, for later use, fix a T_g-invariant probability measure μ_g. In the case where $\langle g \rangle$ is finite, say of order m, we let T_g be the cyclic permutation on m elements, and we let μ_g be the T_g-invariant probability measure taking the value $\frac{1}{m}$ at each point. Form the set

$$Y_g = \prod_{i \in I_g} X_g$$

and topologize it by giving it product topology, hence, Y_g is a Cantor set. (Recall that all Cantor sets are homeomorphic.) Define

$$\alpha^g : G \longrightarrow Homeo(Y_g)$$

by, for $\gamma \in G$,

$$(\alpha^g_\gamma \omega)(j) = T^n_g(\omega(i)),$$

where i and n are uniquely determined by the condition

$$\gamma^{-1}\gamma_j = \gamma_i g^n.$$

It is straightforward to verify that α^g_γ is well-defined and is a homeomorphism on Y_g. In addition we have the following:

Lemma 1.3 *α^g is an action of G on Y_g.*

Proof Let γ, γ' be any pair in G, and ω any element in Y_g. We must show that

$$(\alpha^g_\gamma \circ \alpha^g_{\gamma'})(\omega)(j) = (\alpha^g_{\gamma\gamma'}\omega)(j)$$

for all $j \in I_g$. We have that $\gamma^{-1}\gamma_j = \gamma_i g^m$ and ${\gamma'}^{-1}\gamma_i = \gamma_k g^n$, for some $i, k \in I_g$ and $m, n \in \mathbb{Z}$. This implies that

$$\begin{aligned}(\gamma\gamma')^{-1}\gamma_j &= {\gamma'}^{-1}(\gamma^{-1}\gamma_j)\\ &= {\gamma'}^{-1}\gamma_i g^m\\ &= \gamma_k g^{n+m}.\end{aligned}$$

Hence,

$$\begin{aligned}(\alpha^g_\gamma \circ \alpha^g_{\gamma'})(\omega)(j) &= (\alpha^g_\gamma(\alpha^g_{\gamma'}\omega))(j)\\ &= T^m_g((\alpha^g_{\gamma'}\omega)(i))\\ &= T^m_g(T^n_g(\omega(k)))\\ &= T^{m+n}_g(\omega(k)),\end{aligned}$$

and

$$(\alpha^g_{\gamma\gamma'}\omega)(j) = T^{m+n}_g(\omega(k)).$$

By the choice of T_g in the periodic and non-periodic case, respectively, this is well-defined, and we get the desired equality. □

Define $X = \prod_{g\in G} Y_g$, and the action $\alpha : G \longrightarrow Homeo(X)$ by $\alpha = \prod_{g\in G} \alpha^g$. Also, let μ be the probability measure $\mu = \prod_{g\in G}(\prod_{i\in I_g} \mu_g)$ on X. Then we have all the necessary ingredients to prove Theorem 1.1.

Proof of Theorem 1.1. It is clear that α is an action of G on X. We prove that the action is free. Let $\gamma \in G$, $x \in X$, such that $\alpha_\gamma(x) = x$, and suppose $\gamma \neq e$. Then, in particular, $\alpha^\gamma_\gamma(x(\gamma)) = x(\gamma)$. Let $\omega = x(\gamma) \in Y_\gamma$. By assumption $\alpha^\gamma_\gamma(\omega) = \omega$, and especially $(\alpha^\gamma_\gamma(\omega))(0) = \omega(0)$. By definition of α^γ_γ we have that $(\alpha^\gamma_\gamma(\omega))(0) = T^{-1}_\gamma(\omega(0))$, since $\gamma^{-1}\gamma_0 = \gamma_0\gamma^{-1}$, where γ_0 by our above notation denotes e. Hence, $T^{-1}_\gamma(\omega(0)) = \omega(0)$, and so $T_\gamma(\omega(0)) = \omega(0)$. By construction, T_γ has no fixed points, and so we get a contradiction, thus proving that the action is free.
We now turn to showing that μ is G-invariant, and, hence, that $M(X, G)$ is non-empty. It is enough to consider $\nu_g = \prod_{i\in I_g} \mu_g$ under the action of G on Y_g. In fact, if ν_g is invariant for every $g \in G$, then so is the product measure $\mu = \prod_{g\in G} \nu_g$ on X – this is easily verified. So let $g \in G$, and $A \subset Y_g$ be a Borel rectangle such that A projects to proper Borel subsets of X_g on a finite number of coordinate places, say on $\{i_1, i_2, ...i_k\} \subset I_g$. That is, $\pi_{i_m}(A) = A_{i_m}$, for some Borel subset $A_{i_m} \subsetneq X_g$, where π_i is the projection to the i-th copy of X_g, and $m = 1, 2, ..., k$. Otherwise, when $i \notin \{i_1, i_2, ...i_k\}$, $\pi_i(A) = X_g$. Fix $\gamma \in G$. Then $(\alpha^g_\gamma)^{-1}(A)$

will again be a Borel rectangle such that $(\alpha_\gamma^g)^{-1}(A)$ at coordinate position j_m is $T^{-k_m}A_{i_m}$ according to the equation $\gamma^{-1}\gamma_{i_m} = \gamma_{j_m} g^{k_m}$, where $m = 1, 2, ..., k$. Hence, by T_g-invariance of μ_g, we have

$$\nu_g((\alpha_\gamma^g)^{-1}(A)) = \prod_{m=1}^{k} \mu_g(T^{-k_{i_m}} A_{i_m}) = \prod_{m=1}^{k} \mu_g(A_{i_m}) = \nu_g(A).$$

This proves the result. □

Remark 1.4 First we observe that, for an aperiodic $g \in G$, the choice of (X_g, T_g) can be made quite arbitrary. The proof rests on the fact that T_g has no fixed points. In fact, whenever g is aperiodic, we may choose $X_g = \{0, 1\}$ (discrete topology), (which of course is not Cantor), with T_g defined by $T(0) = 1$ and $T(1) = 0$, and where μ_g is such that $\mu_g(\{0\}) = \mu_g(\{1\}) = \frac{1}{2}$. The space X that we construct in Theorem 1.1 will still be a Cantor set. However, to tackle the still open problem whether there exist a free, uniquely ergodic action, it may potentially be useful to have at ones disposal Cantor minimal systems with special properties, e.g. uniquely ergodic with minimal self-joinings. (Confer Proposition 1.8.)

Corollary 1.5 *For every countable and discrete group G there exist a free and minimal action of G on the Cantor set.*

Proof Let (X, G, α) be a free action of G on the Cantor set X according to Theorem 1.1. Choose, by a Zorns lemma argument, a closed G-invariant subset Z of X such that the restriction of the action of G on Z is minimal. This subaction gives the result, since freeness is obviously preserved, and freeness and minimality of the action implies that Z is again a Cantor set. □

Remark 1.6 The existence of a G-invariant probability measure μ on the large space X, guaranteed by the construction in Theorem 1.1 might be "lost" when moving to a minimal subaction. This is the case when Z is "thin" in X (i.e. $\mu(Z) = 0$). However, if G is amenable there does exist a G-invariant probability measure on Z, (but this is not necessarily induced by the measure μ in Theorem 1.1).

Let G be given and X be as defined above the proof of Theorem 1.1. Let, for every $g \in G$, $\pi_g : X \longrightarrow Y_g$ be the projection onto the component Y_g determined by g. Also, we let $\pi_i : Y_g \longrightarrow X_g$ be the projection to the i-th copy of X_g in Y_g, and define $\pi_i^g : X \longrightarrow X_g$ by $\pi_i^g = \pi_i \circ \pi_g$. We have the following lemma.

Lemma 1.7 *Let G be a countable discrete group and (X, G, α) as constructed above. Fix $g \in G$ and $i \in I_g$ and put $\gamma = \gamma_i g^{-1} \gamma_i^{-1}$. Then the following diagram* *commutes*

$$\begin{array}{ccc} X & \xrightarrow{\alpha_\gamma} & X \\ \downarrow{\scriptstyle \pi_g} & & \downarrow{\scriptstyle \pi_g} \\ Y_g & \xrightarrow{\alpha^g_\gamma} & Y_g \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_i} \\ X_g & \xrightarrow{T_g} & X_g \end{array}$$

Proof The top square commutes by definition of the action α. For the lower square we have that $\pi_i \circ \alpha^g_\gamma(\omega) = \alpha^g_\gamma(\omega)(i) = T_g(\omega(i))$ since $\gamma^{-1}\gamma_i = \gamma_i g \gamma_i^{-1}\gamma_i = \gamma_i g$. This proves the result. $\square$

We will now restrict our discussion to the case where the systems (X_g, T_g) used to build X are uniquely ergodic for every $g \in G$ (i.e. X_g has a unique T_g-invariant probability measure) – say with invariant measure μ_g. That is, $M(X_g, T_g) = \{\mu_g\}$ for all $g \in G$. Also, given a probability measure ν on X we define the *marginals* of ν to be $\{(\pi^g_i)^*\nu\}_{g\in G, i\in I_g}$.

Proposition 1.8 *Let ν be a G-invariant probability measure associated to (X, G, α), where (X, G, α) is constructed as above with uniquely ergodic (X_g, T_g) for every $g \in G$. Then the marginals of ν are $\{\mu_g\}_{g\in G, i\in I_g}$, where $M(X_g, T_g) = \{\mu_g\}$. That is, $(\pi^g_i)^*\nu = \mu_g$, for every $g \in G$ – independent of $i \in I_g$.*

Proof Fix some $g \in G$ and $i \in I_g$. Let A be a Borel subset of X_g. Setting $\gamma = \gamma_i g^{-1} \gamma_i^{-1}$, we have by Lemma 1.7 that

$$\begin{aligned} [(\pi^g_i)^*(\nu)](T_g^{-1}A) &= \nu((\pi^g_i)^{-1}T_g^{-1}A) \\ &= \nu((T_g\pi^g_i)^{-1}A) \\ &= \nu((\pi^g_i\alpha_\gamma)^{-1}A) \\ &= \nu(\alpha_\gamma^{-1}(\pi^g_i)^{-1}A) \\ &= \nu((\pi^g_i)^{-1}A) \\ &= [(\pi^g_i)^*(\nu)](A). \end{aligned}$$

Hence, $(\pi^g_i)^*\nu$ is T_g-invariant on X_g, and so, by unique ergodicity of (X_g, T_g) we must have that $(\pi^g_i)^*\nu = \mu_g$. $\square$

We would like to construct a free continuous action of any countable group G on the Cantor set X that is uniquely ergodic, i.e. $M(X, G)$ consists of one element. However, this remains to be done and Proposition 1.8 is as far as we got at this point.

We end this section with a concrete example of a free action which we will need in the next section. It is not difficult to show that the construction described in this example falls under the general scheme outlined in the proof of Theorem 1.1.

Example 1.9 *Fix a countable group G and $g \in G$ with $g \neq e$. We first consider the Bernoulli shift of G on $\{0,1,2\}^G$, so that for $h, h_0 \in G$ and $f \in \{0,1,2\}^G$ we define $h \cdot f$ by*

$$h \cdot f(h_0) = f(h^{-1}h_0).$$

We let X_g be the set of all $f : G \to \{0,1,2\}$ which has

$$f(h) \neq f(hg)$$

all g. This set is closed and invariant, and it can be quickly seen to be non-empty – note that in the case $\langle g \rangle$ has odd order, we really need to consider $f : G \to \{0,1,2\}$ rather than just $f : G \to \{0,1\}$. For any $f \in X_g$ we have

$$g \cdot f(e) = f(g) \neq f(e)$$

$$\therefore g \cdot f \neq f,$$

and g acts freely on X_g.

Thus by passing to

$$X = \prod_{g \neq e} X_g$$

we obtain a zero dimensional compact space on which G acts freely. We finish by passing to some minimal

$$K \subset X.$$

In conclusion G acts continuously, freely and minimally on K.

2 Minimal non-homogenous actions on compacta

We want to prove the following:

Theorem 2.1 *Every countable group acts continuously and freely on some compact non-homogeneous metric space all of whose orbits are dense.*

We say that a topological space is *non-homogeneous* if the entire homeomorphism group does not act transitively. Recall that an action is *minimal* if every orbit is dense. We will use the generic term G-space for an action (X, G, α).

For the remainder of this section we fix a countable group G and K a compact, zero dimensional, minimal G-space according to Example 1.9. We let $\{x_n : n \in \mathbb{N}\}$ enumerate one of the orbits.

We are going to somehow extend the action of G on K to an action on a closed subset $C \subset K \times [-2,2] = \{(x,y) : x \in K, y \in \mathbb{R}, -2 \leq y \leq 2\}$. Above each $x \in K$ we will add a small fibre in C; most of the time the fibre will be trivial, but above any of the x_n's there will be a small arc segment. The length of the arc segment above x_n will go to zero as $n \to \infty$.

We prove Theorem 2.1 through several steps - starting with the following result.

Lemma 2.2 *There exists continuous $f_1 : K \setminus \{x_1\} \to \mathbb{R}^{>0}$ such that*

(a) $f_1(x) \to \infty$ as $x \to x_1$;

(b) for all $c \in [-1,1], \epsilon > 0, \delta > 0$ there exists x with $d(x, x_1) < \delta$ and

$$|\sin(f_1(x)) - c| < \delta.$$

Proof Partition $K \setminus \{x_1\}$ into non-empty clopen pieces, $(A_n)_{n\in\mathbb{N}}$, with each $A_n \subset \{y : d(y, x_1) < 2^{-n}\}$; note we can do this since the assumptions on the action entail x_1 non-isolated. Let $\{q_n : n \in \mathbb{N}\}$ enumerate $\mathbb{Q}$. At each n choose $a_n \in A_n, b_n \in \mathbb{R}^{>0}$ so that

$$|\sin(\frac{1}{d(a_n, x) + b_n}) - q_n| < 2^{-n}.$$

Because of the way in which we partitioned the space, the function

$$\psi : K \setminus \{x_1\} \to \mathbb{R}^{>0}$$

$$x \mapsto \text{the } b_n \text{ s.t } x \in A_n$$

is continuous, and hence so is

$$f_1 : K \setminus \{x_1\} \to \mathbb{R}^{>0}$$

$$x \mapsto \psi(x) + \frac{1}{d(a_n, x)}$$

is continuous, and by our choices of the a_n's, b_n's exactly as required by the lemma. □

At each n we let $g_n \in G$ be the unique element of G with $g_n \cdot x_1 = x_n$. We then define for each n a continuous

$$f_n : K \setminus \{x_n\} \to \mathbb{R}^{>0}$$

by

$$f_n(x) = f_1(g_n^{-1}(x)).$$

Note for future reference that if $g \in G$ has $g \cdot x_\ell = x_m$, then $g = g_m g_\ell^{-1}$, and hence for all $x \neq x_\ell$

$$f_\ell(x) = f_1(g_\ell^{-1} \cdot x) = f_1(g_m^{-1} g_m g_\ell^{-1} \cdot x)$$

$$= f_m(g_m g_\ell^{-1} \cdot x) = f_m(g \cdot x).$$

Moreover we again have at each $c \in [-1,1], \epsilon > 0, \delta > 0$ some x with $d(x, x_1) < \delta$ and $|\sin(f_n(x)) - c| < \delta$.

Now to describe the fibres. Above each $x \notin \{x_n : n \in \mathbb{N}\}$ we add just the point (x, y) where

$$y = \sum_{n\in\mathbb{N}} 2^{-n} \sin(f_n(x)).$$

Above points of the form x_n we add all (x_n, y) where

$$y = 2^{-n}w + \sum_{m \neq n} 2^{-m}\sin(f_m(x))$$

for some w with $-1 \leq w \leq 1$.

Now to define the action. For $x \notin \{x_n : n \in \mathbb{N}\}$ and $(x, y) \in C$ we let $g \cdot (x, y) = (g \cdot x, z)$ where z is the unique point with $(g \cdot x, z) \in C$. On the other hand, for $x = x_n$ and $y = 2^{-n}w + \sum_{m \neq n} 2^{-m}\sin(f_m(x_n))$ we let $g \cdot (x_n, y) = (x_\ell, z)$ where $g \cdot x_n = x_\ell$ and

$$z = 2^{-\ell}w + \sum_{m \neq \ell} 2^{-m}\sin(f_m(x_\ell)).$$

Lemma 2.3 *The resulting G-action is continuous.*

Proof Fix $g \in G$, $(r_i, y_i)_i$ in C, with $(r_i, y_i) \to (r, y)$. We want to show that $g \cdot (r_i, y_i) \to g \cdot (r, y)$. We may confine our discussion to the cases that either every $r_i \in \{x_n : n \in \mathbb{N}\}$ or every $r_i \notin \{x_n : n \in \mathbb{N}\}$. After that there is a further split into subcases depending on the circumstances of r.

We fix $(z_i)_i, z$ so that each $g \cdot (r_i, y_i) = (g \cdot r_i, z_i)$ and $g \cdot (r, y) = (g \cdot r, z)$.

Case(1) Each $r_i \notin \{x_n : n \in \mathbb{N}\}$.
Subcase(1a) $r \notin \{x_n : n \in \mathbb{N}\}$.

Then since each f_m is continuous on $K \setminus \{x_n : n \in \mathbb{N}\}$ and since $g \cdot r_i \to g \cdot r$,

$$z_i = \sum_{m \in \mathbb{N}} 2^{-m}\sin(f_m(g \cdot r_i)) \to z = \sum_{m \in \mathbb{N}} 2^{-m}\sin(f_m(g \cdot r))$$

$$\therefore (g \cdot r_i, z_i) \to (g \cdot r, z).$$

Subcase(1b) $r = x_n$, some n.

Let $g \cdot x_n = x_\ell$. We suppose $y = 2^{-n}w + \sum_{m \neq n} 2^{-m}\sin(f_m(x_n))$, and hence $y = 2^{-\ell}w + \sum_{m \neq \ell} 2^{-m}\sin(f_m(x_n))$. Since $y_i \to y$ we have

$$\sum_m 2^{-m}\sin(f_m(r_i)) \to 2^{-n}w \sum_{m \neq n} 2^{-m}\sin(f_m(x_n));$$

and then since $r_i \to x_n$ and each f_m is continuous away from x_m,

$$\sin(f_m(r_i)) \to \sin(f_m(x_n))$$

for $m \neq n$. Putting these statements together we have

$$2^{-n}\sin(f_n(r_i)) \to 2^{-n}w.$$

Recalling the equivariance assumptions on the functions $\{f_n : n \in \mathbb{N}\}$ we have therefore

$$2^{-\ell}\sin(f_\ell(g \cdot r_i)) = 2^{-\ell}\sin(f_n(r_i)) \to 2^{-\ell}w.$$

For $m \neq \ell$ we immediately obtain

$$2^{-m}\sin(f_m(g \cdot r_i)) \to 2^{-m}\sin(f_m(x_\ell))$$

since $g \cdot r_i \to g \cdot x_n = x_\ell$ and each f_m is continuous away from x_m. Hence

$$z_i = \sum_m 2^{-m}\sin(f_m(g \cdot r_i)) \to x_\ell = 2^{-\ell}w + \sum_{m \neq \ell} 2^{-m}\sin(f_m(g \cdot r)),$$

which completes this subcase.

Case(2) Each $r_i \in \{x_n : n \in \mathbb{N}\}$. Let $r_i = x_{n(i)}$ and

$$y_i = 2^{-n(i)}w_i + \sum_{m \neq n(i)} 2^{-m}\sin(f_m(x_{n(i)}).$$

We let $\hat{n}(i)$ be defined by $g \cdot x_{n(i)} = x_{\hat{n}(i)}$. We may replace each r_i by $s_i \notin \{x_n : n \in \mathbb{N}\}$ where each s_i is close enough to $x_{n(i)}$ to ensure not only do we have

$$d(x_{n(i)}, s_i) < 2^{-i},$$

$$|\sum_{m \neq n(i)} 2^{-m}\sin(f_m(s_i)) - \sum_{m \neq n(i)} 2^{-m}\sin(f_m(x_{n(i)}))| < 2^{-i},$$

but also

$$d(x_{\hat{n}(i)}, g \cdot s_i) < 2^{-i},$$

$$|\sum_{m \neq \hat{n}(i)} 2^{-m}\sin(f_m(g \cdot s_i)) - \sum_{m \neq \hat{n}(i)} 2^{-m}\sin(f_m(x_{\hat{n}(i)}))| < 2^{-i};$$

appealing to the assumptions on $(f_n)_n$ we can do this so that we additionally have

$$|\sin(f_{n(i)}(s_i)) - w_i| < 2^{-i},$$

and therefore by the invariance properties of these functions

$$|\sin(f_{\hat{n}(i)}(g \cdot s_i)) - w_i| < 2^{-i}.$$

At each i we let

$$u_i = \sum_m 2^{-m}\sin(f_m(s_i)),$$

$$v_i = \sum_m 2^{-m}\sin(f_m(g \cdot s_i)).$$

We have set things up so that $(s_i, u_i) \to (r, y)$, $g \cdot (s_i, u_i) = (g \cdot s_i, v_i)$ and if $(g \cdot s_i, v_i)$ has a limit then that point is the limit of $g \cdot (x_{n(i)}, y_i)$. But now case(1) indeed gives that since $(s_i, u_i) \to (x, r)$ we have

$$(g \cdot s_i, v_i) \to g \cdot (x, r),$$

thereby completing the proof in case(2), and thus the proof of the lemma. □

Lemma 2.4 *Every orbit is dense.*

Proof Fix $(x,y),(x',y')\in C$. We want to find $(g_{n(i)})_i$ with

$$g_{n(i)}\cdot(x,y)\to(x',y').$$

Case(1) Both x and x' lie outside $\{x_n : n\in\mathbb{N}\}$.

This follows more or less automatically.

Case(2) $x\notin\{x_n : n\in\mathbb{N}\}$ and $x'=x_n$ for some n.

We will have

$$y'=2^{-n}w'+\sum_{m\neq n}2^{-m}\sin(f_m(x_n)),$$

some $w'\in[-1,1]$. Then as we saw in the proof of the last lemma, at each $\ell\in\mathbb{N}$ we can find $r_\ell\notin\{x_n : n\in\mathbb{N}\}$ such that

$$|\sum_{m\neq n}\sin(f_m(x_n))-\sum_{m\neq n}\sin(f_m(r_\ell))|<2^{-\ell-1}$$

$$|w'-\sin(f_n(r_\ell))|<2^{-\ell-1},$$

and hence for

$$y_\ell=\sum_m 2^{-m}\sin(f_m(x_\ell))$$

we have $|y_\ell-y'|<2^{-\ell}$. Since case(1) shows that each (x_ℓ,y_ℓ) is in the closure of the orbit of (x,y), and since $(x_\ell,y_\ell)\to(x',y')$, we have that (x',y') is in the closure of the orbit of (x,y).

Case(3)

We finally consider the case of $x\in\{x_n : n\in\mathbb{N}\}$. In light of case(2), we may assume $x'\notin\{x_n : n\in\mathbb{N}\}$. Fix $\delta>0$; we want to show some $g\cdot(x,y)=(z,\hat{y})$ has $d(z,x')<\delta$ and $|\hat{y}-y'|<\delta$. By assumptions on the G-space K, we may find $g\in G$ such that

$$d(g\cdot x,x')<\delta,$$

$$g\cdot x\notin\{x_m : m>\log_2(\frac{2}{\delta})+1\},$$

and at each $m<\log_2(\frac{2}{\delta})$

$$|\sin(f_m(g\cdot x))-\sin(f_m(x'))|<\frac{\delta}{2}.$$

From this it follows that if $g\cdot x=x_n$ then

$$|\sum_{m<n}2^{-m}\sin(f_m(x_n))-\sum_{m<n}2^{-m}\sin(f_m(x'))|<\sum_{m<n}2^{-m}\frac{\delta}{2}<\frac{\delta}{2},$$

whilst

$$\sum_{m\geq n} 2^{-m} = 2^{-n+1} < \frac{\delta}{2}.$$

Hence since

$$y' = \sum_m 2^{-m}\sin(f_m(x'))$$

we have $|\hat{y} - y'| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$. □

Lemma 2.5 *K is compact.*

Proof We began with

$$K_0 = \{(x, \sum_m 2^{-m}\sin(f_m(x))) : x \in C\},$$

which is compact in virtue of being the graph of a continuous function on a compact set. Then we added a sequence of compact sets of the form

$$B_n = \{x_n\} \times I_n,$$

for some closed interval I_n, where $|I_n| \to 0$ and each B_n meets K_0.

In general this kind of construction keeps us inside the class of compact sets. □

Lemma 2.6 *C is not homogeneous.*

Proof Let $(U_n)_n$ be a basis consisting of clopen set. Note then that for any $x \notin \{x_n : n \in \mathbb{N}\}$

$$\{U_m \times [-2, 2] \cap C : x \in U_m\}$$

provides a neighborhood basis at x consisting of clopen sets.

On the other hand, at any of the x_n's no such neighborhood basis is possible since there is a homeomorphic copy of the unit interval inside C passing through x. □

3 Amenability

We want to explore different characterization of amenable groups. Amenability can be defined in several different (equivalent) ways. In the countable case it can be stated as follows: A countable group G is amenable if, for any continuous action of G on a compact, metrizable space X, there exist a probability measure μ on X which is invariant by G (i.e. $\mu(gE) = \mu(E)$ for any Borel set $E \subset X$). Giordano and de la Harpe proves in [4] that for a countable group G to be amenable it is sufficient that any continuous action of G on the Cantor set has an invariant probability measure. With Theorem 1.1 established we are now in position to give the following characterization:

Proposition 3.1 *A countable group G is amenable if and only if for any minimal and free action of G on the Cantor set, there exist a G-invariant probability measure on X.*

Proof The "only if" part is obvious. Now for the "if" part. First we establish that, for G to be amenable, it is sufficient that there exists a G-invariant probability measure on the Cantor set for minimal actions. We do this by showing that whenever a G-invariant probability measure exists for minimal actions then this is also true in the general case. So suppose G is an action on the Cantor set X. By a Zorn's lemma argument there exists a closed, G-invariant subset Y of X, such that G acts minimally on Y. Let μ_Y be a G-invariant probability measure on Y, and extend this to a probability measure μ_X on X by

$$\mu_X(E) = \mu_Y(E \cap Y)$$

for a Borel set $E \subset X$. Now, Y and $X \backslash Y$ are G-invariant, hence

$$\mu_X(gE) = \mu_Y((gE) \cap Y) = \mu_Y((g(E \cap Y)) = \mu_Y(E \cap Y) = \mu_X(E),$$

and so μ_X is a G-invariant probability measure on X.

Now let G be an action on the Cantor set X. By using Theorem 1.1, pick a free action of G on the Cantor set Z, and define an action of G on $X \times Z$ by $g(x, z) = (gx, gz)$ for $g \in G$. Clearly this yields a free action on the Cantor set (a product of Cantor sets is still Cantor), and hence, by our assumption, there exist a G-invariant probability measure μ on $X \times Z$. Let $\pi : X \times Z \longrightarrow X$ be the projection $(x, z) \longmapsto x$, and define the probability measure ν on X by $\nu = \pi^*(\mu)$ (i.e. $\nu(E) = \mu(\pi^{-1}(E))$ for a Borel set $E \subset X$). By construction of the action of G on $X \times Z$ we have that, for all $g \in G$, $\pi \circ g = g \circ \pi$. This means that

$$\nu(gE) = \mu(\pi^{-1}(gE)) = \mu(g\pi^{-1}(E)) = \mu(\pi^{-1}(E)) = \nu(E),$$

for all $g \in G$ and any Borel set E in X. Hence ν is a G-invariant measure on X, and so we are done. □

4 A topological version of the Furman and Adams counterexample

In the measure-theoretic setting Furman [3] (Theorem D) proved that there exists countable non-singular equivalence relations that can not be generated by an essentially free action of some countable group. Adams [1] showed an analogous result in the Borel setting. We will now turn to a construction yielding a topological version of this.

Proposition 4.1 *There is a continuous action of a countable group Γ on a zero-dimensional compact space such that*

(i) every orbit is dense;

(ii) there is an invariant subspace X_0 which carries a Γ-invariant probability measure and for which $E_\Gamma|_{X_0}$ is induced by the action of an amenable group;

(iii) there is a subspace X_1 for which $E_\Gamma|_{X_1}$ admits an invariant measure μ with the property that $E_\Gamma|_{X_1}$ is not μ-invariant in the sense of [9].

A consequence from this is an answer to a question raised in [5]:

Corollary 4.2 *There is a countable group Γ acting continuously and minimally on Cantor space such that the associated orbit equivalence relation E_Γ is not induced by any continuous (or even Borel)* free *action of a countable group Δ.*

The point is that the group Δ would have to be amenable, since it has a free action on a probability space X_0 with amenable equivalence relation; but then E_Γ would be "1-amenable" in the sense of [6], and in particular so would X_1, since 1-amenability goes down from an equivalence relation to its sections, and in particular $E_\Gamma|_{X_1}$ would have to be amenable relative to any measure on the space.

Remark 4.3 By using Adams approach it is an easy consequence of Theorem 1.1 to give an example of an equivalence relation, with countable equivalence classes, that cannot be freely generated. However, the orbits in this equivalence relation split into disjoint, nonempty clopen sets in the underlying space, and, hence, is far from being minimal.

The rest of the text is devoted to proving Proposition 4.1.

In what follows, $\mathbb{F}_2 = \langle a, b\rangle$ is the free group generated by the elements a and b. For ψ in the homeomorphism group of a compact space metric (K, d), $\|\psi\|$ refers to the sup norm metric, $\sup_{x\in K} d(x, \psi(x))$. In particular the canonical Polish topology is given by the complete metric $D(\psi, \phi) = \|\psi\circ\phi^{-1}\| + \|\phi\circ\psi^{-1}\|$.

Lemma 4.4 *There is a locally finite countable group G acting continuously on a perfect, compact, zero-dimensional space, C_0,*

$$G \to \mathrm{Hom}(C_0),$$

$$g \mapsto \psi_g,$$

and there is a continuous action of $\mathbb{F}_2$ on C_0,

$$\mathbb{F}_2 \to \mathrm{Hom}(C_0),$$

$$\sigma \mapsto \psi_s\sigma,$$

such that

(i) every orbit is dense under the $\mathbb{F}_2$ action and every orbit is dense under the G action;

(ii) for $c \in \{a, b\}$ there is a sequence $(c_n)_{n\in\mathbb{N}}$ in G such that

$$\|\psi_c \circ \psi_{c_n}^{-1}\| \to 0,$$
$$\|\psi_{c_n} \circ \psi_c^{-1}\| \to 0;$$

(iii) there is a probability measure μ_0 on C_0 which is simultaneously invariant under both actions.

Proof Let H_n be a decreasing sequence of finite index normal subgroups $\mathbb{F}_2$ such that for all $\sigma \in \mathbb{F}_2$, $\sigma \neq e$, there is n with $\sigma \notin H_n$. At each n let X_n be the space of (left) cosets of H_n. $\mathbb{F}_n$ acts in the obvious way on X_n. We let C_0 be the usual profinite completion – so that $C_0 \subset \prod X_n$ consists of functions

$$f : \mathbb{N} \to X$$

with each $f(n) \in X_n$ and $f(n+1) \subset f(n)$. We equip this with the product topological structure and the product action $(\sigma \cdot f)(n) = \sigma \cdot (f(n))$. Note that indeed every orbit is dense in this action, and that C_0 is compact, zero-dimensional, and without isolated, and hence homeomorphic to Cantor space; therefore it suffices to find an action of a countable G on C_0 such that for each $\sigma \in \mathbb{F}_2$ there exist (g_n) in G with

$$\sup_{f\in C_0}(d_{C_0}(\sigma \cdot f, g_n \cdot f) + d_{C_0}(\sigma^{-1} \cdot f, g_n^{-1} \cdot f)) \to 0.$$

(Here we let $d_{C_0}(f_1, f_2)$ be the reciprocal of the least n with $f_1(n) \neq f_2(n)$.)

For each n and $s \in \prod_{i\leq n} \mathbb{F}_2/H_i$ with $s(i+1) \subset s(i)$, let $V_s = \{f \in C_0 : f \supset s\}$. Then the collection of all such V_s's form a basis for the topology. Note that we may find homeomorphisms $\pi_{s,t} : V_s \to V_t$ for s, t of the same length which are all arranged so that

(i) $\pi_{s,t} \circ \pi_{u,s} = \pi_{u,t}$ when s, t, u all of the same length;

(ii) if $s' \supset s, t' \supset t$ then $\pi_{s,t}|_{V_{s'}} = \pi_{s',t'}$.

We then define a subgroup G_n of $\mathrm{Hom}(C_0)$ at each n generated by a_n, b_n defined by the specifications

(i) $a_n \cdot V_s = V_t$ if and only if $a \cdot V_s = V_t$;

(ii) $b_n \cdot V_s = V_t$ if and only if $b \cdot V_s = V_t$;

(iii) in the event of $a_n \cdot V_s = V_t$ we have $a_n|_{V_s} = \pi_{s,t}$;

(iv) in the event of $b_n \cdot V_s = V_t$ we have $b_n|_{V_s} = \pi_{s,t}$.

In other words, a_n, b_n approximate the homeomorphisms associated to a and b by behaving as they do at the first n-levels of the space, but then having a purely "flat" behavior from that stage onwards.

It is easily checked that these provide the required the group of homeomorphism. We get the invariant measure by first instance noting that both these actions preserve a compatible metric, and in general the isometry group of a compact metric space is compact and in particular amenable. □

We now let $\hat{G} = G * \mathbb{F}_2$. We define a sequence of actions of $\hat{G}$ on the Cantor space C_0 above. For $n \in \mathbb{N}$ we let α_n be the action given by the above indicated action of G and for $\mathbb{F}_2$ we take the action suggested by identifying a with a_n, b with b_n; in other words, for $g \in G$

$$\alpha_n(g, x) = \psi_g(x),$$

and for $c \in \{a, b\}$

$$\alpha_n(c, x) = \psi_{c_n}(x);$$

by the definition of free product, this extends uniquely to an action of the free product $G * \mathbb{F}_2$.

We also define an action α_∞ by $\alpha_\infty(\sigma, x) = \psi_\sigma(x)$ for $\sigma \in \mathbb{F}_2$ and $\alpha_\infty(g, x) = \psi_g(x)$ for $g \in G$.

Let $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$, the group of addition modulo 3. We let $\prod_{\mathbb{N}} \mathbb{Z}_3$ be the compact group obtained by the countable product of $\mathbb{Z}_3$; we let $\mathbb{Z}_3^{<\mathbb{N}}$ be the subgroup of elements with finite support. Note that the smaller, countable group acts by left translation on the larger, preserves the Haar measure, and has all orbits dense. Let $\beta : \mathbb{Z}_3^{<\mathbb{N}} \times \prod \mathbb{Z}_3 \to \prod \mathbb{Z}_3$ be this action.

We let $\Gamma = \hat{G} * \mathbb{Z}_3^{<\mathbb{N}}$. We define actions of $\mathbb{Z}_3$ and $\hat{G}$ on $C_0 \times \prod \mathbb{Z}_3$, which will in turn induce an action of the free product Γ.

The action of $\mathbb{Z}_3^{<\mathbb{N}}$ is easy to describe. $\tau \cdot (x, f) = (x, \beta(\tau, f))$. All the work is in the second coordinate, where we take the translation action of the countable group on the compact group from which it is hewn.

For $\hat{G}$ the situation is more complicated; there is a split in cases. First of all, if f has *no* n with $f(n) = \bar{0}$ (the identity of the 3 element group) then we let $\sigma \cdot (x, f) = (\alpha_\infty(\sigma, x), f)$. If there is some n with $f(n) = \bar{0}$ then we go to the first such n and let $\sigma \cdot (x, f) = (\alpha_n(\sigma, x), f)$.

Lemma 4.5 *The action of* Γ *on* $C_0 \times \prod \mathbb{Z}_3$ *is continuous.*

Proof Define $\rho : C_0 \to \mathbb{N} \cup \{\infty\}$, the one point compactification of $\mathbb{N}$, by $\rho(f) =$ least n with $f(n) = \bar{0}$ (infinity if no such n). It is easily seen that this function is continuous and for any $f_n \to f$ the homeomorphisms

$$C_0 \to C_0$$

$$x \mapsto \alpha_{f_n}(\sigma, x)$$

approach the homeomorphism

$$x \mapsto \alpha_f(\sigma, x)$$

in the sup norm metric. □

We then let μ_1 be Haar measure on $\mathbb{Z}^{<\mathbb{N}}$. We then consider the measure $\mu_0 \times \mu_1$ on $C = C_0 \times \prod \mathbb{Z}_3$. The measure concentrates on pairs (x, f) where f has infinitely many n's with $f(n) = \bar{0}$. On this invariant subspace the equivalence relation E_Γ is induced by the amenable group $G \times \mathbb{Z}_3^{<\mathbb{N}}$.

On the other hand, for any f with no n with $f(n) = \bar{0}$ we let consider the closed subspace $C_0 \times \{f\} \subset C$. The equivalence relation on this slice admits the invariant measure obtained from μ_0 and includes the orbit equivalence relation induced by a free action of $\mathbb{F}_2$, and is hence not amenable relative to μ_0.

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