Yet another proof of Gaboriau-Popa

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Abstract

This note recasts the proof of Gaboriau-Popa in a way which does not use property (T).

1 Preface and clarification

In [1], after a number of partial results by various authors, Damien Gaboriau and Sorin Popa showed that non-abelian countable free groups have uncountably many measure preserving, ergodic, actions on standard Borel probability spaces up to orbit equivalence. The original proof used the language and ideas of operator algebras and in some sense was driven by a subtle application of $\mathbb{Z}\rtimes\mathrm{SL}_2(\mathbb{Z})$ having relative property (T) over $\mathbb{Z}^2$. In [3] the proof was reorganized, in some ways refined, and, although the ultimate ideas drew their inspiration from the operator algebraic techniques of [1], written entirely in *spatial* language, avoiding any mention of concepts specific to the field of operator algebras.

The point of this brief remark is to go one step further and give a proof which eliminates any use of property (T).

However the title of this note is already something of a misnomer, since at the conceptual level there is serious ambiguity regarding *whether this is actually a new proof.* The key point is to finesse out of an appeal to property (T), but what actually happens below is that critical idea in proving that $\mathbb{Z}\rtimes\mathrm{SL}_2(\mathbb{Z})$ has relative property (T) is applied directly into the proof of Gaboriau-Popa. As such, this is not so much a new proof, but rather a radically different way of looking at the preexisting argument.

The main step in the note below is to isolate a dynamical notion of group actions, which for lack of a better word I called *expansive.*

2 Proof

Definition Let (X, d) be a separable, complete metric space. Let Γ be a countable group. A Borel action of Γ on (X, d) is *expansive* if there exists $\epsilon > 0$ and Borel $A_1, ..., A_n \subset X \times X$ such that:

1. $\bigcup_{i\leq n} A_i \supset \{(x, y) \in X^2 : d(x, y) < \epsilon\}$; and
2. for each A_i there exists $F_i \subset \Gamma$ such that

$$\forall j \neq i(F_i \cdot A_i \cap A_j \subset \Delta(X) =_{df} \{(x, x) : x \in X\}$$

and

$$\bigcap_{\gamma\in F_i} \gamma \cdot A_i \subset \Delta(X).$$

(Here we are thinking of Γ acting on X^2 via the diagonal action: $\gamma \cdot (x, y) = (\gamma \cdot x, \gamma \cdot y)$.)

Lemma 2.1 *Let Γ act on (X, d) in an expansive fashion. Then any almost invariant Borel probability measure on X^2 which largely concentrates on $\{(x, y) \in X^2 : d(x, y) < \epsilon\}$ largely concentrates on $\Delta(X)$.*

To be more precise, for $A_1, ..., A_n, F_1, ..., F_n$ as above, for each $\delta_0 > 0$ there exists a $\delta_1 > 0$ such that if μ is Borel probability measure on X^2 with

$$\mu(\{(x,y) \in X^2 : d(x,y) < \epsilon\}) > 1 - \delta_1$$

and

$$|\mu(\gamma \cdot A_i) - \mu(A_i)| < \delta_1$$

all $i \leq n, \gamma \in A_i$, then

$$\mu(\Delta(X)) > 1 - \delta_0.$$

Proof This is simply a result of unpacking the definitions. □

Lemma 2.2 *Let (X, d) be a separable, complete, metric space. Let μ be an atomless Borel probability measure on X. Let Γ be a countable group acting in an ergodic, measure preserving, and expansive fashion.*

Let $(E_s)_{0<s<t}$ be countable Borel, measure preserving, equivalence relations with:

1. *for $s \neq t$, $A \subset X$ of positive measure, we have at a.e. $x \in A$*

$$A \cap [x]_{E_s} \neq A \cap [x]_{E_t};$$

2. *each $E_s \supset E_\Gamma$.*

Then each E_s is orbit equivalent to only countably many E_t's.

Proof Assuming otherwise, we will start winding towards a contradiction.

Fix the $A_1, ..., A_n$ and $F_1, ..., F_n$ as in the definition of expansive. Let $F = \bigcup_{i \leq n} F_i$.

Fix some $t_0 \in (0, 1)$ for which there is $W \subset (0, 1)$ with

$$|W| = \aleph_1$$

and such that for each $s \in W$ we have some fixed

$$\theta_s : X \to X$$

witnessing E_{t_0} orbit equivalent to E_s.

Let Δ be a countable group acting in a Borel manner on X such that

$$E_\Delta = E_{t_0}.$$

Keep in mind that the action of Δ will be measure preserving. Fix some well ordering of F^{-1}. For $s \in W$ define

$$\varphi_s : X \times \Delta \to F^{-1} \cup \{\infty\}$$

by letting $\varphi(x, \delta)$ be the least γ (in the well ordering of F^{-1}) such that

$$\gamma \cdot \theta_s(x) = \theta_(\delta \cdot x),$$

if such a γ exists, and equal to ∞ otherwise.

We the let Y_0 be the collection of measurable functions from $X \times \Gamma$ to Δ with

$$d(\varphi, \varphi') = \sum_{\gamma \in F^{-1}} \mu(\{x : \exists \delta (\varphi_{s_1}(x, \delta) = \gamma \Leftrightarrow \varphi_{s_2}(x, \delta) \neq \gamma)\}).$$

Let Y_1 be the collection of measurable functions from X to X with

$$d_1(\theta, \theta') = \int d(\theta(x), \theta'(x)) d\mu(x).$$

From the way the definitions have been set up it follows that $d_1(\theta, \theta')$ equals $\sum_{\delta \in \Delta} \mu(\{x : \varphi_{s_1}(x, \delta) = \varphi_{s_2}(x, \delta)\})$.

After the customary identification of functions agreeing a.e, (Y_0, d_0) and (Y_1, d_1) both become separable metric spaces. W is uncountable so we can find $s_1, s_2 \in W$ with

$$s_1 \neq s_2,$$

$$d_0(\varphi_{s_1}, \varphi_{s_2}) \sim 0,$$
$$d_1(\theta_{s_1}, \theta_{s_2}) \sim 0.$$

Let

$$\theta = (\theta_{s_1}, \theta_{s_2}) : X \to X^2,$$
$$x \mapsto (\theta_{s_1}(x), \theta_{s_2}(x)).$$

Let

$$\nu = \theta^*[\mu]$$

– i.e. $\nu(A) = \mu(\theta^{-1}[A])$.

Claim: For $\gamma \in F$

$$\gamma \cdot \nu \sim \nu$$

– i.e. for every Borel $C \subset X^2$ we have $\nu(\gamma^{-1}(C)) \sim \nu(C)$.

Proof of Claim: Fix such C and let $D = \theta^{-1}[C]$. Let

$$D_\delta = \{y \in D : \theta(\delta \cdot y) = \gamma^{-1}\theta(y)\}.$$

Observe that

$$D_\delta = \{y \in D : \varphi_{s_1}(y, \delta) = \varphi_{s_2}(y, \delta) = \gamma\},$$

and thus

$$\mu(D \setminus \bigcup_\delta D_\delta) \sim 0,$$

by $d_0(\varphi_{s_1}, \varphi_{s_2}) \sim 0$. From this we can derive that

$$\nu(\theta[D \setminus \bigcup_\delta D_\delta]) = \nu(\theta[D] \setminus \theta[\bigcup_\delta D_\delta]) \sim 0.$$

Then

$$\gamma^{-1} \cdot \theta[D_\delta] = \theta[\delta \cdot D_\delta]$$

And from there we have

$$\gamma \cdot \nu(\theta[D_\delta) = \nu(\gamma^{-1} \cdot \theta[D_\delta]) = \nu(\theta[\delta \cdot D_\delta])$$
$$= \mu(\delta \cdot D_\delta) = \mu(D_\delta),$$

since the Δ action is measure preserving. This shows then that

$$\nu(\bigcup_\delta \theta[D_\delta]) = \gamma\nu(\bigcup_\delta \theta[D_\delta]).$$

Since $\nu(\theta[D] \setminus \theta[\bigcup_\delta D_\delta]) \sim 0$ the claim is proved. (Claim$\Box$)

But then this implies that ν is a Borel probability measure on X^2 with $|\nu(\gamma \cdot A_i) - \nu(A_i)| \sim 0$ all $i \leq n, \gamma \in A_i$. $d_1(\theta_{s_1}, \theta_{s_2}) \sim 0$ yields $\theta(y) \in \{(x, x') : d(x, x') < \epsilon\}$ for all but a very small μ-measure set of $y \in X$, which in turn yields $\nu(\{(x, x') : d(x, x') < \epsilon\}) \sim 1$.

Thus by applying the last lemma we have that ν largely concentrates on the diagonal $\Delta(X)$. So we can find some $B \subset X$ of positive measure with

$$\theta[B] =_{df} \{(\theta_{s_1}(y), \theta_{s_2}(y)) : y \in B\} \subset \Delta(X).$$

Letting $A = \theta_{s_1}[B]$ we have

$$\theta_{s_1}^{-1}|_A = \theta_{s_2}^{-1}|_A.$$

Since θ_{s_1} and θ_{s_2} effect orbit equivalences, we then have that for a.e. $x \in A$

$$A \cap [x]_{E_{s_1}} = A \cap [x]_{E_{s_2}},$$

with a contradiction to the assumptions on the equivalence relations $(E_s)_{s\in(0,1)}$. □

The remainder of Gaboriau-Popa, using the refinements of [3], requires five further facts.

Lemma 2.3 *Let*

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

Then A, B freely generates a copy of $\mathbb{F}_2$ in $SL_2(\mathbb{Z})$.

This is a basic fact in combinatorial group theory. It suffices to see that reduced words of the form

$$A^{k_{2n-1}}B^{k_{2n-2}}..B^{k_2}A^{k_1}$$

or

$$B^{k_{2n}}A^{k_{2n-1}}B^{k_{2n-2}}..B^{k_2}A^{k_1}$$

are never equal to the identity. Here an induction on n shows that in the first case we have the entry in first row second column of greater absolute value than all the other entries, while in the other case the entry in the second row second column is dominant.

Notation From now on let A and B be given their above meanings in the special linear group $\mathrm{SL}_2(\mathbb{Z})$. Let $\Gamma = \langle A, B\rangle$. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and take the canonical action of of Γ induced as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 2.4 *The action of Γ on $\mathbb{T}^2$ is expansive.*

Taking $n = 4$ and

$$A_1 = \{((s,t),(s',t') : 0 \le s - s' \le \frac{1}{100}, 0 \le t - t' \le \frac{1}{100}\},$$

$$A_2 = \{((s,t),(s',t') : -\frac{1}{100} \le s - s' \le 0, 0 \le t - t' \le \frac{1}{100}\},$$

$$A_3 = \{((s,t),(s',t') : 0 \le s - s' \le \frac{1}{100}, -\frac{1}{100} \le t - t' \le 0\},$$

$$A_4 = \{((s,t),(s',t') : -\frac{1}{100} \le s - s' \le 0, -\frac{1}{100} \le t - t' \le 0\}$$

with

$$F_1 = \{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\},$$

$$F_2 = \{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\},$$

and so on. The point here is that inside a small neighborhood of $\Delta(\mathbb{T}^2)$ the diagonal action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2 \times \mathbb{T}^2$ resembles the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^2$. This is exactly the point which arises in showing $\mathbb{Z}\rtimes\mathrm{SL}_2(\mathbb{Z})$ has relative (T) over $\mathbb{Z}^2$.

Definition Let m denote Lebesgue measure on $\mathbb{T}^2$. Let $M_\infty(\mathbb{T}^2, m)$ denote the group of measure preserving automorphisms of $(\mathbb{T}^2, m)$ with the *weak topology*, generated by subbasic open sets of the form

$$\{\varphi : (\varphi[C]\Delta B) < \epsilon\}.$$

It is a well known fact that if we undertake the usual identification of functions agreeing a.e. then this forms a Polish group. See for instance [2].

Lemma 2.5 *(Törnquist, [3]) The collection of $\varphi \in M_\infty(\mathbb{T}^2, m)$ such that A, B, φ freely generates a free group is comeager.*

Lemma 2.6 *(See [1]) Given $\varphi \in M_\infty(\mathbb{T}^2, m)$, we can find an array $(\varphi_s)_{s\in(0,1)}$ of elements in $M_\infty(\mathbb{T}^2, m)$ such that at all $s < t$ we have that for a.e. $x \in \mathbb{T}^2$ that*

$$\{\varphi_s^\ell(x) : \ell \in \mathbb{Z}\} \subsetneq \{\varphi_t^\ell(x) : \ell \in \mathbb{Z}\} \subsetneq \{\varphi^\ell(x) : \ell \in \mathbb{Z}\}.$$

Applying the last lemma to the lemma before last, we obtain a family $(\varphi_s)_{s\in(0,1)}$ such that each $\langle A, B, \varphi_s\rangle$ freely generates a copy of $\mathbb{F}_3$. Then if we let E_s be the resulting orbit equivalence relation on $\mathbb{T}^2$, then each E_s is induced by a measure preserving action of $\mathbb{F}_3$, and by 2.2 we have each E_s is orbit equivalent to only countably many other E_t's.

3 Queries

Having gone to all the trouble of isolating this dynamical notion of *expansive* in order to eliminate the Hilbert space theoretic notion of *property (T)*, it is natural to wonder if these two are somehow related.

Question Let Γ be a countable group having relative property (T) over an infinite, normal, abelian, subgroup Δ. Does Γ/Δ admit an expansive action by measure preserving automorphisms on a compact abelian group?

It seems rather fantastic to suppose that relative property (T) would imply some kind of dynamical property for a group's actions, but the hope for a positive answer would be based around looking at the induced action of Γ/Δ on $\hat{\Delta}$. I would not be shocked if there is at least *some* kind of result along those lines, but possibly involving a weakening of the notion of expansive.

References

[1] D. Gaboriau, S. Popa *An uncountable family of nonorbit equivalent actions of* $\mathbb{F}_n$, **J. Amer. Math. Soc.** 18 (2005), no. 3, 547–559

[2] A.S. Kechris, **Classical descriptive set theory,** Graduate Texts in Mathematics 156, Springer-Verlag, New York, 1995.

[3] A. Törnquist, *Orbit equivalence and actions of* $\mathbb{F}_n$, **J. Symbolic Logic,** 71 (2006), no. 1, 265–282.