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RIGIDITY AND EQUIVALENCE RELATIONS WITH INFINITELY MANY ENDS

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Abstract. We consider groups and equivalence relations with infinitely many ends and the problem of selecting one end in a uniform manner. In general a non-amenable equivalence relation may have infinitely many ends and yet admit a Borel function selecting from each class a single end; however, we show that in the presence of an invariant Borel probability measure, the equivalence having infinitely many ends precludes the existence of a Borel method of selecting an end from each class.

By analyzing the action of a group on its space of ends, we obtain a new proof of an earlier superrigidity result due to Monod and Shalom for actions of certain product groups.

Finally, we discuss applications of equivalence relations with infinitely many ends to the theory of percolation and to the abstract theory of Borel equivalence relations. In particular, we show that for equivalence relations with infinitely many ends, all the amenable normal subequivalence relations are smooth.

1. Introduction

Informally, given a graph $\mathcal{G} = (V, E)$ we can form the *space of ends* on that graph. An end will be a consistent assignment of connected components in the induced subgraph $\mathcal{G} \setminus F$ to each finite set of vertices $F \subseteq V$, so that distinct finite sets are always assigned non-disjoint components. A particular case is when a group itself has infinitely many ends in the structure provided by a Cayley graph.

The following result is implicit in Theorem 2.20, Chapter II of [DD89] (though given its parallels with the later selection theorems for equivalence relations we go to the trouble of giving an independent proof in Section 2).

Theorem 1.1. *(Dicks-Dunwoody [DD89] and also [Dun97]) Let Γ be a countable group. Assume that with respect to some finite generating set Γ has infinitely many ends. Then:*

(i) the stabilizer of each finite subset of the ends of size greater than two will be finite;

(ii) the stabilizer of a pair of ends will be amenable;

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(iii) the stabilizer of a single end will have infinite index in the group, though it is not necessarily amenable.

In terms of the resulting equivalence relation arising from Γ acting on its ends, we are unable to say much when Γ has a single end. However, looking at the action of finite sets of ends, the arguments from the proof of Theorem 1.1 show the following:

Theorem 1.2. *Let Γ be a countable group. Assume that with respect to some finite generating set Γ has infinitely many ends. Consider the natural action by translation of the group on the space of ends and finite sets of ends.*

(i) The orbit equivalence relation on pairs of ends is amenable.

(ii) The orbit equivalence relation on subsets of size at least three is smooth.

Recall, from say [Kec95], that an equivalence relation is *smooth* if there is a Borel function assigning a real number to each equivalence class as a complete invariant. In the context of an equivalence relations arising from the Borel action of a countable group, an equivalent definition of smoothness is that there be a Borel set meeting each orbit in exactly one point. In Section 3 we use 1.2 to give a new proof of a special case of a superrigidity theorem proved previously by Nicolas Monod and Yehuda Shalom [MS06].

The case of equivalence relations with many ends has been previously considered by Miller [Mil], who obtained results parallel to the ones in Theorem 1.2.

Theorem 1.3. *(Miller) Let E be a countable Borel equivalence relation on a Polish space X. Assume $\mathcal{R}$ is a Borel graphing on X whose connected components form the E-classes. Assume that each class has infinitely many ends in this graphing. Then:*

(i) if one can in a Borel way assign exactly two ends to each class, then the equivalence relation is hyperfinite;

(ii) if one can in a Borel way assign a finite set of size at least 3 to each class, then the equivalence relation is smooth.

This left open the situation where exactly one end can be assigned to each class, and we observe later in the paper that this is not in general sufficient to imply hyperfiniteness or even amenability of the equivalence relation. Furthermore, in light of 1.1 (iii) one might be cautious about drawing any conclusion from the assumption that one can select an end. Nonetheless, in the measure theoretic context we can say the following:

Theorem 1.4. *Let E be a countable, Borel, measure preserving equivalence relation on a standard Borel probability space (X, μ). If E admits a graphing with infinitely many ends, then one cannot choose a single end for each class in a Borel manner.*

From this, we can obtain a corollary for normal subequivalence relations. $F \subset E$ is said to be *normal* if there is a group action generating E which respects F-equivalence, i.e., xFx' implies $g \cdot xFg \cdot x'$ for every element g of the group.

Theorem 1.5. *Let E be a countable Borel measure preserving equivalence relation on a standard Borel probability space. If E admits a locally finite graphing with infinitely many ends, then every normal subequivalence relation of E is either smooth or non-amenable.*

This in turn gives an application to percolation.

Theorem 1.6. *Let Γ be a finitely generated non-amenable group. Let Δ be a normal subgroup. Suppose S is a finite generating set for Γ with a non-unique percolation phase on the resulting Cayley graph. Suppose we choose $p \in (p_c, p_u)$ in the non-uniqueness phase.*

Then almost surely a randomly chosen percolation on the Cayley graph with the indicated measure (of leaving in each edge independently with probability exactly p) will have all its infinite clusters meeting Δ in only finitely many points.

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2. Groups with infinitely many ends

2.1. **Background on groups and equivalence relations.** In this section, we recall some basic notions concerning countable Borel equivalence relations. For much more on the subject, we recommend the book by Kechris and Miller [KM04].

Considering a standard Borel space X, an equivalence relation E on X is *countable* if every E-class is countable. In particular, if Γ is a countable group acting by Borel automorphisms on X, then the orbit equivalence relation that arises from this action is countable.

A group Γ is *amenable* if for every finite $Q \subset \Gamma$ and $\epsilon > 0$, there is a non-negative function $f \in \ell^1(\Gamma)$ with norm 1 and such that for every $\gamma \in Q$,

$$\|\gamma \cdot f - f\|_{\ell^1(\Gamma)} < \epsilon$$

or, equivalently, if there is a finite subset $F \subset \Gamma$ such that for every $\gamma \in Q$,

$$|\gamma \cdot F \Delta F| < \epsilon \, |F| \, .$$

Similarly, an equivalence relation E is *amenable* if there is a sequence $\lambda^n \colon E \to \mathbb{R}$ of non-negative Borel functions such that

(1) $\lambda^n_x \in \ell^1([x]_E)$ where $\lambda^n_x(y) = \lambda^n(x,y)$ for xEy;
(2) $\|\lambda^n_x\|_{\ell^1([x]_E)} = 1$;
(3) for all xEy, $\left\|\lambda^n_x - \lambda^n_y\right\|_{\ell^1([x]_E)} \to 0$.

Furthermore, E is *smooth* if there is a Borel map $\phi\colon X \to \mathbb{R}$ such that

$$xEy \iff \phi(x) = \phi(y).$$

Note that all finite Borel equivalence relations are smooth since it is possible to just select a point from every equivalence class in a Borel way.

In the presence of a probability measure μ on X, we have the notions of *smooth a.e.* or *amenable a.e.*. Moreover, an action of Γ on X is *measure-preserving* if for every Borel $A \subseteq X$,

$$\mu(A) = \mu(\gamma \cdot A)$$

and *ergodic* if whenever a Borel subset $A \subseteq X$ is invariant under the action of Γ, then A must be null or conull.

The *Cayley graph* $C(\Gamma, S)$ of a group Γ with respect to a symmetric generating set is formed by letting the set of vertices be Γ and letting the set of edges be defined by

$$E = \big\{(\gamma s, \gamma) \mid \gamma \in \Gamma, s \in S\big\}.$$

A Borel *graphing* of an equivalence relation E is a Borel subset $\mathcal{G} \subseteq X^2$ such that the connected components of $\mathcal{G}$ correspond to the equivalence classes in E. As an example, an orbit equivalence relation given by a group action can always be graphed by

$$\mathcal{G} = \big\{(s \cdot x, x) \mid x \in X, s \in S\big\}.$$

In the remainder of the paper, we will be considering countable groups that admit Cayley graphs with infinitely many ends as well as countable equivalence relations that admit graphings with infinitely many ends.

2.2. **The space of ends.** We introduce the formal notion of the space of ends of a group as well as a topology so that the space becomes compact metrizable.

Let $G = (V, E)$ be a connected graph. An *end* is a function φ with domain consisting of finite connected subsets of V such that

(1) for each finite connected $F \subseteq V$, $\varphi(F)$ is a connected component of $V \setminus F$;
(2) for $F_1, F_2 \subseteq V$ finite connected, $\varphi(F_1) \cap \varphi(F_2) \neq \emptyset$.

For $F \subseteq V$ finite and connected, let C_F be the set of components of $V \setminus F$. If C_F is infinite, let C_F^* be $C_F \cup \{\infty_F\}$ equipped with the topology where the open sets are the finite subsets of C_F and the cofinite subsets of C_F^*. Then each C_F^* is compact with respect to this topology. On the other hand, if C_F is finite, let $C_F^* = C_F$ equipped with the discrete topology.

We let the *space of ends* $E(G)$ be the collection of all ends in G with the product topology: A basic open set has the form

$$\{\phi \in E(G) : \varphi(F_1) = C_1, \varphi(F_2) = C_2, ..., \varphi(F_n) = C_n, \}$$

for $F_1, F_2, ..., F_n$ a finite sequence of finite connected sets in V such that $C_i \in C_{F_i}$ for every $i \leq k$.

Similarly, the *extended ends* $E^*(G)$ is the collection of all functions from finite connected sets such that at each F, $\varphi(S) \in C_F^*$. Again we take the product topology on this space.

In [Mil] the notion of end is explored through the natural equivalence relation on *rays*, which we include here for the reader's convenience. The definition we have chosen is different but equivalent to the definition given in [Mil]. The main appeal of the definition as presented here is in terms of making the appropriate topology self-evident. In the case that V is countable, $E(G)$ is a Polish space.

Let $G = (V, E)$ be a connected graph. A *ray* in G is a one-to-one function from $\mathbb{N}$ to G. Given two rays $\vec{\alpha} = (\alpha_0, \alpha_1, ...), \vec{\beta} = (\beta_0, \beta_1, ...)$ we say they are *end equivalent* if for every finite set $F \subset V$, there is a path from $\vec{\alpha}$ to $\vec{\beta}$ avoiding F.

Each ray naturally gives rise to an end in our sense. Given a ray $\vec{\alpha}$, we let $\varphi_{\vec{\alpha}}$ be defined by

$$\varphi_{\vec{\alpha}}(F) = C$$

if and only if for all sufficiently large n we have

$$\alpha_n \in C.$$

Since any ray is one to one, it eventually leaves each finite set and then will spend the remainder of its journey inside one of the components of $V \setminus F$. It is easily seen that $\vec{\alpha}$ and $\vec{\beta}$ are end equivalent if $\varphi_{\vec{\alpha}} = \varphi_{\vec{\beta}}$. Moreover for any end φ we can find a ray $\vec{\alpha}$ with $\varphi_{\vec{\alpha}} = \varphi$:- First choose $\vec{\beta} \in V^{\mathbb{N}}$ such that given any finite connected F we eventually have $\beta_n \in \varphi(F)$; then contract the function β to a ray α by excising all the repeated vertices and the edges in between.

Given a set of generators S for a countable group Γ, we denote by $E(\Gamma, S)$ the set of ends of $C(\Gamma, S)$. Let $E(\Gamma, S)^*$ (the extended ends) be the set of maps from finite subsets $F \subseteq V$ to C^*_F that satisfy the following conditions:

(1) If $F_1, F_2 \subseteq V$ are finite connected and $\varphi(F_1) \neq \infty_{F_1}$ and $\varphi(F_2) \neq \infty_{F_2}$, then $\varphi(F_1) \cap \varphi(F_2) \neq \emptyset$.
(2) If $\varphi(F_1) = \infty_{F_1}$ and $F_1 \cap F_2 = \emptyset$, then $\varphi(F_2)$ is the component containing F_1.

Remark It follows from the definition of $E(\Gamma, S)^*$ that if $F_1, F_2 \subset V$ and $\varphi \in E(\Gamma, S)^*$, then either $F_1 \subseteq \phi(F_2)$ or $F_2 \subseteq \phi(F_1)$.

Similarly then, if $\varphi \in E(\Gamma, S)$ and F_1, F_2 as above, non-disjointness of $\varphi(F_1)$ and $\varphi(F_2)$ entails $F_1 \subseteq \varphi(F_2)$ or $F_2 \subseteq \varphi(F_1)$.

Proposition 2.1. *$E(\Gamma, S)^*$ with the product topology is a compact metrizable space.*

Proof. Let us consider the space X of maps φ from finite connected subsets $F \subseteq V$ to C^*_F equipped with the product topology. We show that $E(\Gamma, S)^*$ is closed in X with respect to this topology.

Suppose that $\varphi_n \to \varphi$ in the product topology. $F_1, F_2 \subseteq V$ are finite connected such that $\varphi(F_1) \neq \infty_{F_1}$ and $\varphi(F_2) \neq \infty_{F_2}$. Then let $\varphi(F_1) = C_1$ and $\varphi(F_2) = C_2$. There is then some $m \in \mathbb{N}$ such that $\varphi_n(F_1) = C_1$ and $\varphi_n(F_2) = C_2$ for all $n > m$ since

$$\{\varphi \in X \mid \varphi(F_1) = C_1, \varphi(F_2) = C_2\}$$

is an open set in the product topology. But then since $\varphi_n \in E(\Gamma, S)^*$, it must be that

$$\varphi(F_1) \cap \varphi(F_2) = C_1 \cap C_2 = \varphi_n(F_1) \cap \varphi_n(F_2) \neq \emptyset.$$

On the other hand, suppose that $\varphi(F_1) = \infty_{F_1}$ and $F_1 \cap F_2 = \emptyset$. We then claim that $\varphi(F_2)$ is the component containing F_1. Let C be the component in $V \setminus F_1$ containing F_2. Then there is some $m \in \mathbb{N}$ such that $\varphi_n(F_1) \neq C$ for all $n > m$. Then $\varphi_n(F_2)$ is the component containing F_1 for all $n > m$ and $\varphi(F_2) = \varphi_n(F_2)$. This establishes that $E(\Gamma, S)^*$ is closed in X.

Finally, since $E(\Gamma, S)^*$ is a closed subset of the second countable space X, $E(\Gamma, S)^*$ is second countable as well. Moreover, $E(\Gamma, S)^*$ is a zero-dimensional compact space and must be normal. By Urysohn's metrisability theorem, $E(\Gamma, S)^*$ is metrisable.

□

We let $E(\Gamma, S) = E(C(\Gamma, S))$, the space of ends for Γ given S as a generating set. We will let Γ act on $E(\Gamma, S)$ by

$$(g \cdot \varphi)(F) = g \cdot (\varphi(g^{-1} \cdot F)) = \{g \cdot \sigma \mid \sigma \in \varphi(g^{-1} \cdot F)\}$$

where $g \cdot \infty = \infty$. This action is then easily checked to be continuous.

From now on we will be assuming $C(\Gamma, S)$ has infinitely many ends. This in particular implies Γ is non-amenable (see [KM04]).

2.3. **Proof of Theorem 1.1.** First, we reduce the situation to just considering the action of Γ on the space $E(\Gamma, S)$.

Proposition 2.2. *If $\varphi \in E(\Gamma, S)^* \setminus E(\Gamma, S)$, then φ has finite stabiliser.*

Proof. Suppose that $g \cdot \varphi = \varphi$. Then $g \cdot \varphi(F) = \infty_F$. Since

$$g \cdot \varphi(F) = \{g \cdot \sigma \mid \sigma \in \varphi(g^{-1} \cdot F), \}$$

this implies that

$$\varphi(g^{-1} \cdot F) = \infty_{g^{-1} \cdot F}.$$

However, there is a finite subset $F_0 \subseteq \Gamma$ such that for $g \notin F_0$, $g^{-1} \cdot F \cap F = \emptyset$, which then would ensure that $\varphi(g^{-1} \cdot F) \neq \infty_{g^{-1} \cdot F}$. □

To deal with the action on triples and pairs of ends, we will need the following lemma:

Lemma 2.3. *Let φ_1, φ_2 be distinct ends in $E(\Gamma, S)$. Let $\mathcal{F}$ be a collection of finite, connected subsets of Γ such that:*

(1) $F_1 \neq F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 = \emptyset$;

(2) *each* $F \in \mathcal{F}$ *has* $\varphi_1(F) \neq \varphi_2(F)$.

Then if we define R *on* $\mathcal{F}$ *by*

$$F_1 R F_2$$

if $F_1 \neq F_2$ *and* $F_2 \subseteq \varphi_1(F_1)$ *then* R *is a linear ordering on* $\mathcal{F}$.

Proof. Transitivity of R is a routine consequence of the definition of an end. From the fact that either $F_1 \subseteq \phi(F_2)$ or $F_2 \subseteq \phi(F_1)$, we obtain for all $F_1, F_2 \in \mathcal{F}$ that either $F_1 = F_2$ or $F_1 R F_2$ or $F_2 R F_1$.

Finally we need to avoid the situation that $F_2 \subseteq \varphi_1(F_1)$ *and* $F_1 \subseteq \varphi_1(F_2)$ for $F_1, F_2 \in \mathcal{F}$ distinct. But this would entail $F_2 \not\subseteq \varphi_2(F_1)$ and $F_1 \not\subseteq \varphi_2(F_2)$, yielding a contradiction. □

Proposition 2.4. *The stabiliser* $\Gamma_{(\varphi_1,\varphi_2,\varphi_3)}$ *of a triple of distinct ends is finite.*

Proof. For $l \neq k \in \{1,2,3\}$, let $F_{l,k} \subseteq V$ be finite such that $\varphi_l(F) \neq \varphi_k(F)$. Then $F = \bigcup_{l \neq k \in \{1,2,3\}}$ disconnects all three ends simultaneously.

If $g \in \Gamma_{(\varphi_1,\varphi_2,\varphi_3)}$, then

$$\varphi_l(F) = (g \cdot \varphi_l)(F) = g \cdot \varphi_l(g^{-1} \cdot F)$$

and, as a result,

$$\varphi_l(g^{-1} \cdot F) \neq \varphi_k(g^{-1} \cdot F)$$

for $k \neq l \in \{1,2,3\}$. Thus, $F \cap g^{-1} \cdot F \neq \emptyset$ since otherwise by lemma 2.3, we can find some $m \in \{1,2,3\}$ with

$$F \subseteq \varphi_m(g^{-1} \cdot F), \quad g^{-1} \cdot F \subseteq \varphi_m(F).$$

Since $g^{-1} \cdot F \cap F = \emptyset$ except for finitely many g, $\Gamma_{(\varphi_1,\varphi_2,\varphi_3)}$ must be finite. □

Definition For $g, h \in \Gamma$, let $d(g,h)$ be the least $m \in \mathbb{N}$ such that there exist $s_1, s_2, \ldots, s_m \in S$ with

$$s_1 s_2 \cdots s_m g = h.$$

Similarly, for $F, H \subseteq \Gamma$ and $g \in \Gamma$ we let

$$d(g,F) = \inf\{d(g,h) \mid h \in F\},$$
$$d(H,F) = \inf\{d(\sigma,\tau) : \sigma \in H, \tau \in F\}.$$

Proposition 2.5. *The stabiliser* $\Gamma_{(\varphi_1,\varphi_2)}$ *of a pair of distinct ends is amenable.*

Proof. Fix some finite connected F containing the identity of the group which disconnects the ends, i.e.,

$$\varphi_1(F) \neq \varphi_2(F).$$

Let $\mathcal{G} = \{g \cdot F : g \in \Gamma_{(\varphi_1,\varphi_2)}\}$. Note that each element of $\mathcal{G}$ also disconnects φ_1, φ_2. Let $\mathcal{G}_n$ be the elements F' of $\mathcal{G}$ for which there exists some $m \leq n$ and $s_1, s_2, ..., s_m \in S$ with

$$s_1 s_2 \cdots s_m \in F'/$$

In other words, these are the elements of $\mathcal{G}$ whose distance from e in the Cayley graph is at most n.

Lemma 2.5.1. *There does not exist a set of three disjoint elements of* $\mathcal{G}_{n+1} \setminus \mathcal{G}_n$.

Proof. Otherwise suppose F_1, F_2, F_3 were disjoint elements of this set. We may define R as in Lemma 2.3 by HRH' if $H' \subseteq \varphi_1(H), H \cap H' = \emptyset$, and appealing again to Lemma 2.3 we have that F_1, F_2, F_3 are linearly ordered under R. Without loss of generality, $F_1 R F_2, F_2 R F_3$.

Then $e \notin \varphi_2(F_2)$ would imply

$$d(e, F_1) > d(e, F_2),$$

with a contradiction to the assumptions on F_1, F_2. But then we obtain $e \notin \varphi_1(F_2)$, and another contradiction with

$$d(e, F_3) > d(e, F_2).$$

□

Let K be the cardinality of the set $\{F' \in \mathcal{G} : F' \cap F \neq \emptyset\}$. The setup of our situation gives for all other $F_0 \in \mathcal{G}$ we again have

$$K = |\{F' \in \mathcal{G} : F' \cap F_0 \neq \emptyset\}|.$$

Appealing to the claim we have

$$|\bigcup \mathcal{G}_{n+1} \setminus \bigcup \mathcal{G}_n| \leq 2K.$$

Moreover for each $g \in \Gamma_{(\varphi_1, \varphi_2)}$ with $d(e, g) \leq m$ we have

$$g \cdot \mathcal{G}_n \subseteq \mathcal{G}_{m+n}.$$

Looking at this information we can go ahead and let

$$H_n = \Gamma_{(\varphi_1, \varphi_2)} \cap (\bigcup \mathcal{G}_n)$$

at each n. For $g \in \Gamma_{(\varphi_1, \varphi_2)}$ we have some fixed M, depending on $d(e, g)$ such that at every n

$$|H_n \Delta g \cdot H_n| \leq M,$$

ensuring the H_n's form a Folner set for $\Gamma_{(\varphi_1, \varphi_2)}$. □

Remark In general, the stabilizer of a single end *need not* be amenable. Let $\Gamma_1 \cong \mathbb{F}_2 \times \mathbb{Z}$ and let $\Gamma_2 \cong \mathbb{F}_2$ and let $\Lambda = \Gamma_1 * \Gamma_2$ (with the obvious generating set derived from the canonical generators for $\mathbb{Z}$ and $\mathbb{F}_2$). Then for each $\sigma \in \Gamma_2$ we obtain an end which roughly corresponds to $\sigma \cdot \Gamma_1$ and has stabilizer $\sigma \cdot \Gamma_1 \cdot \sigma^{-1}$.

Proposition 2.6. *Let* $\varphi \in E(\Gamma, S)$*. Then the index of its stabilizer* Γ_φ *is infinite in* Γ.

Proof. We split into cases:

Case(i): There exists a finite connected $F \subseteq G$ and $g \in \Gamma_\varphi$ with

$$F \not\subseteq \varphi(g \cdot F),$$

$$F \cap g \cdot F = \emptyset.$$

We fix such an F and let $\mathcal{A} = \{g \cdot F \mid g \in \Gamma_\varphi\}$. For each $F_0 \in \mathcal{A}$ we let

$$\mathcal{A}_n(F_0) = \{F' \in \mathcal{A} \mid F_0 \not\subseteq \varphi(F'), F_0 \cap F' = \emptyset, d(F_0, F') \leq n\}.$$

We also let

$$\mathcal{A}^+(F_0) = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n(F_0) = \{F' \in \mathcal{A} \mid F_0 \not\subseteq \varphi(F'), F_0 \cap F' = \emptyset\}.$$

Lemma 2.6.1. *If $F', F'' \in \mathcal{A}^+(F_0)$ then either*

(1) *$F' \cap F \neq \emptyset$; or*
(2) *any path from F' to F_0 intersects F'' and $F' \subseteq \varphi(F'')$; or*
(3) *any path from F'' to F_0 intersects F' and $F'' \subseteq \varphi(F')$.*

Proof. Assume $F' \cap F'' = \emptyset$. Then $F' \subseteq \varphi(F'')$ or $F'' \subseteq \varphi(F')$. We may as well assume $F' \subseteq \varphi(F'')$. Let p_0 be a path from F'' to F' inside $\varphi(F'')$.

For a contradiction let p_1 be a path from F' to F_0 avoiding F''. Let p be a path inside F'' from the last point of p_0 to the first point in p_1. Then the concatenation

$$p_0 {}^\frown p {}^\frown p_1$$

gives us a path in $\varphi(F'')$ from F'' to F_0, contradicting $F'' \in \mathcal{A}^+(F_0)$. □

Lemma 2.6.2. *If F', F'' are disjoint elements in $\mathcal{A}^+(F_0)$ with*

$$F' \subseteq \varphi(F'')$$

and any path from F_0 to F' intersecting F'' (i.e. we are in case (ii) of Lemma 2.6.1), then

$$F' \in \mathcal{A}^+(F'').$$

Proof. Otherwise let p_0 be a path from F' to F'' inside $\varphi(F')$. Let p_1 be a shortest path from F'' to F_0 – and note that the assumptions of the claim yield p_1 avoiding F'.

Then letting p being a path inside F'' from the head of p_0 to the tail of p_1 we get

$$p_0 {}^\frown p {}^\frown p_1$$

witnessing $F_0 \subseteq \varphi(F')$, with a contradiction to the assumption $F' \in \mathcal{A}^+(F_0)$. □

Let Λ be the set of $g \in \Gamma_\varphi$ with $g \cdot F \cap F = \emptyset$. Finitely many translates of Λ cover Γ_φ.

For each $n \in \mathbb{N}, F_0 \in \mathcal{A}$ let

$$H_n(F_0) = \bigcup \mathcal{A}_n(F_0).$$

Lemma 2.6.3. *If $g \in \Gamma_\varphi$, $g \cdot F \in \mathcal{A}^+(F)$, then there is a fixed M such that*

$$|\mathcal{A}_m(F) \Delta \mathcal{A}_m(g \cdot F)| \leq M$$

all m.

Proof. Let $d(F, g \cdot F) = n$. If $d(F, F') > n, F' \in \mathcal{A}(F), g \cdot F \cap F' = \emptyset$, then $d(F, g \cdot F) < d(F, F')$, and the last two lemmas yield

$$F' \in \mathcal{A}(g \cdot F).$$

Thus

$$\mathcal{A}_m(F) \setminus \mathcal{A}_m(g \cdot F) \subseteq (\bigcup \mathcal{A}_n(F)) \cup \{F' \in \mathcal{A} : F' \cap g \cdot F \neq \emptyset\},$$

which has a fixed bound to its cardinality. Since $|\mathcal{A}_m(g \cdot F)| = |\mathcal{A}_m(F)|$ we get a bound on $|\mathcal{A}_m(F) \Delta \mathcal{A}_m(g \cdot F)|$ as well. □

Lemma 2.6.4. *If $g \in \Gamma_\varphi$, $F \in \mathcal{A}^+(g \cdot F)$ then there is a fixed M such that*

$$|\mathcal{A}_m(F) \Delta \mathcal{A}_m(g \cdot F)| \leq M$$

at every m.

Proof. By the proof of the last lemma, interchanging F with $g \cdot F$, g with g^{-1}. □

Lemma 2.6.5. *If $g \in \Gamma_\varphi, F \subseteq \varphi(g \cdot F), g \cdot F \subseteq \varphi(F)$, then there is a fixed M such that*

$$|\mathcal{A}_m(F) \Delta \mathcal{A}_m(g \cdot F)| \leq M$$

at every m.

Proof. Say $d(F, g \cdot F) = n$. Then at every m

$$\mathcal{A}_m(F) \subseteq \mathcal{A}_{m+n}(g \cdot F)$$

and so it suffices to show there is a fixed bound on each

$$|\mathcal{A}_{k+1}(g \cdot F) \setminus \mathcal{A}_k(g \cdot F).$$

But it follows from Lemma 2.6.1 that if $F', F'' \in \mathcal{A}_{k+1}(g \cdot F) \setminus \mathcal{A}_k(g \cdot F)$ then $F' \cap F'' \neq \emptyset$. Moreover since Γ_φ acts so as to preserve all the relevant structure we have for any $F' \in \mathcal{A}^+(g \cdot F)$ the set

$$\{F'' \in \mathcal{A}^+(g \cdot F) : F'' \cap F' \neq \emptyset\}$$

is bounded in cardinality by

$$\{h \cdot F : h \in \Gamma, h \cdot F \cap F \neq \emptyset\}.$$

□

Putting these lemmas together we have for every $g \in \Lambda$, $g \cdot F \cap F = \emptyset$, and we will be in one of the cases described by the last three lemmas, and hence we will have a fixed bound on

$$|\mathcal{A}_m(F) \Delta \mathcal{A}_m(g \cdot F)|.$$

Even in the case $g \in \Gamma_\varphi$ we also obtain a bound. First of all given arbitrary $g \in \Gamma_\varphi$ we can assume Γ_φ infinite, and then take $g' \in \Gamma_\varphi$ with

$$F \cap g'F, \quad gF \cap g'F$$

both empty. Then using the bounds on

$$|\mathcal{A}_m(F) \Delta \mathcal{A}_m(g'F)|$$

and

$$|\mathcal{A}_m(gF)\Delta\mathcal{A}_m(g'F)|$$

we obtain a bound on

$$|\mathcal{A}_m(F)\Delta\mathcal{A}_m(g'F)|.$$

Now we can choose some $\sigma \in F$ and let

$$H_m = \Gamma_\varphi \cap (\bigcup \mathcal{A}_m(F)\sigma^{-1}) = \Gamma_\varphi \cap \bigcup \{F' \cdot \sigma^{-1} : F' \in \mathcal{A}_m(F)\}.$$

By the earlier lemmas, for any $g \in \Gamma_\varphi$ we always have a fixed bound on

$$|\bigcup \mathcal{A}_m(F)\Delta \bigcup \mathcal{A}_m(g \cdot F)|,$$

and hence on

$$|\bigcup \mathcal{A}_m(F)\sigma^{-1}\Delta \bigcup \mathcal{A}_m(g \cdot F)\sigma^{-1}|.$$

Since $g \cdot \mathcal{A}_m(F) = \mathcal{A}_m(g \cdot F)$ we obtain a likewise bound on

$$|H_m \Delta g \cdot H_m|.$$

We have $H_m \subset H_{m+1}$ and so it suffices to show $\bigcup_m H_m$ is infinite.

We successively choose g_i's in Λ and define F_i's such that

$$F_1 =_{df} g_1 \cdot F_0 \in \mathcal{A}^+(F_0),$$

$$F_{i+1} =_{df} g_{i+1} \cdot F_i \in \mathcal{A}^+(F_i),$$

which we can always do since Λ acts by automorphisms on the Cayley graph and fixes φ. Then each $F_i \in \mathcal{A}^+(F)$, and

$$g_i g_{i-1} \cdot g_1 \in g_i g_{i-1} \cdots g_1 F_0 \sigma^{-1} = F_i \sigma^{-1},$$

which in turn is included in

$$\bigcup_{m \in \mathbb{N}} H_m.$$

Case(ii): For all $F \subseteq G$ finite connected, $g \in \Gamma_\varphi$,

$$F \cap g \cdot F = \emptyset \Rightarrow F \subseteq \varphi(g \cdot F).$$

Replacing g by g^{-1} we as well obtain

$$g \cdot F \subseteq \varphi(F).$$

Fix such an F_0 finite and connected for which there exists some end ψ with

$$\psi(F_0) \neq \varphi(F_0).$$

Assuming Γ_φ to be infinite we let $\mathcal{C}$ be an infinite set of disjoint Γ_φ translates of F_0. For each $F \in \mathcal{C}$ choose an end φ_F with

$$\varphi_F(F) \neq \varphi(F).$$

Lemma 2.6.6. *If $F' \neq F'' \in \mathcal{C}$, then*

$$\varphi_{F'}(F') \cap \varphi_{F''}(F'') = \emptyset.$$

Proof. Instead let σ be an element of $\varphi_{F'}(F') \cap \varphi_{F''}(F'')$ and let p_0 be a shortest path from σ to F'. We can either choose a path from σ to F' which doesn't pass through F'' or a path from σ to F'' which doesn't pass through F' – let us assume the former case. But then we have $F' \subseteq \varphi(F'')$ by case assumption, and σ becomes a common element of $\varphi_{F''}(F'')$ and $\varphi(F'')$, contradicting the assumption on $\varphi_{F''}$. □

Lemma 2.6.7. *At each $F \in \mathcal{C}$ we can find $g \in \Gamma$ such that*

$$g \cdot F \cap F = \emptyset,$$
$$g \cdot F \subseteq \varphi_F(F).$$

Proof. Since $\varphi_F(F)$ is infinite, we can find infinitely many g's which have $g \cdot F \cap \varphi_F(F) \neq \emptyset$. Only finitely many of these g's will have $g \cdot F = \emptyset$. □

In light of this claim, at each $F \in \mathcal{C}$ we let $g_F \in \Gamma$ be such that

$$g_F \cdot F \cap F = \emptyset,$$
$$g_F \cdot F \subseteq \varphi_F(F),$$

and, *when possible*,

$$F \not\subseteq (g_F \cdot \varphi)(g_F \cdot F).$$

Subcase (iia): At some $F \in \mathcal{C}$ we have $F \subseteq (g_F \cdot \varphi)(g_F \cdot F)$.

Then let $F_1 = F, g_1 = g_F, \varphi_1 = \varphi$, and applying our subcase assumption successively choose $g_n \in \Gamma$ such that for

$$F_{n+1} = g_n \cdot F_n,$$
$$\varphi_{n+1} = g_n \cdot \varphi_n,$$

we get

$$d(F_{n+1}, F) > d(F_n, F),$$
$$F_{n+1} \not\subseteq \varphi_n(F_n),$$

and

$$F_{n+1} \cap F_m = \emptyset$$

all $m \leq n$.

Now assuming for a contradiction that Γ_φ has finite index in G, we could arrange

$$\varphi_n = \varphi_2$$

and $g_n \in G_{\varphi_2}$ all $n \geq 2$.

This would in particular give

$$g_2 g_1 \cdot F = g_2 g_1 \cdot F_1 = g_2 \cdot F_2 = F_3 \not\subseteq \varphi_2(F_2) = (g_1 \cdot \varphi_1)(g_1 \cdot F)$$
$$\therefore g_1^{-1} g_2 g_1 \cdot F \not\subseteq (g_1^{-1} g_1 \cdot \varphi)(g_1^{-1} g_1 \cdot F) = \varphi(F).$$

But since $g_1^{-1} g_2 g_1 \cdot \varphi = g_1^{-1} g_2 \varphi_2 = g_1^{-1} \varphi_2 = \varphi$ we get $g_1^{-1} g_2 g_1 \in \Gamma_\varphi$ with a contradiction to the assumptions of case (ii).

Subcase (iib): At every $F \in \mathcal{C}$ we have $F \nsubseteq (g_F \cdot \varphi)(g_F \cdot F)$.

Then $(g_F \cdot \varphi)(g_F \cdot F) \subseteq \varphi_F(F)$, and hence we obtain from Lemma 2.6.6 that for F_1, F_2 distinct elements of $\mathcal{C}$

$$(g_{F_1} \cdot \varphi)(g_{F_1} \cdot F_1) \cap (g_{F_2} \cdot \varphi)(g_{F_2} \cdot F_2) = \emptyset.$$

Thus $(g_F \cdot \varphi)_{F \in \mathcal{C}}$ are all distinct ends, showing the stabiliser Γ_φ to indeed have infinite index in Γ. □

In summary, the following hold for the action of Γ on $E(\Gamma, S)$:

(1) $\Gamma \curvearrowright [E(\Gamma, S)^*]^3$ has finite stabilisers.
(2) $\Gamma \curvearrowright [E(\Gamma, S)^*]^2$ has amenable stabilisers.
(3) $\Gamma \curvearrowright E(\Gamma, S)^*$ has stabilisers of infinite index.

and we have completed the proof of Theorem 1.1.

2.4. **Proof of Theorem 1.2.** The proof of the results for Theorem 1.1 gives that the equivalence relation of Γ acting on triples (or larger finite sets) of ends is smooth. In the case of pairs, we obtain amenability.

Proposition 2.7. *The orbit equivalence relation E given by the diagonal action of Γ on distinct pairs of ends is amenable.*

Proof. For every (φ_1, φ_2), there is some finite subset $F \subset V$ such that $\varphi_1(F) \neq \varphi_2(F)$. Let $\{F_m\}_{m \in \mathbb{N}}$ enumerate the finite subsets of V containing the identity element. We then obtain a Borel map

$$F \colon [E(\Gamma, S)]^2 \mapsto \mathbb{N}$$

such that $F(\phi_1, \phi_2)$ disconnects φ_1 and φ_2 by letting $F(\varphi_1, \varphi_2)$ be the least $m \in \mathbb{N}$ such that some Γ translate of F_m disconnects φ_1 and φ_2. Note that F is Borel and E_Γ-invariant since for any $g \in \Gamma$,

$$\varphi_1(F) \neq \varphi_2(F) \iff g \cdot \varphi_1(g \cdot F) \neq g \cdot \varphi_2(g \cdot F).$$

We now define

$$G_n(F) = \{g \cdot F \mid g \in \Gamma, d(e, g \cdot F) \leq n\}.$$

To show that E is amenable, we define a sequence $\lambda^n \colon E \to \mathbb{R}$ as follows:

$$\lambda^n(x, y) = \begin{cases} \frac{1}{|G_n(F(x))|} & \text{if } y = g \cdot x, \quad g \cdot F \in G_n(F(x)) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We now consider the functions $\lambda^n_x \in l^1([x]_E)$ given by $\lambda^n_x(y) = \lambda^n(x, y)$. It is clear that $\|\lambda^n_x\| = 1$ for all $x \in X$. It remains to show that

$$\left\|\lambda^n_x - \lambda^n_y\right\| \to 0$$

as $n \to \infty$ for all xEy.

Let xEy. Then $y = g \cdot x$ for some $g \in \Gamma$. From now on, we will denote $G_n(F(x))$ by G_n since we will only be referring to a single equivalence class of E.

Lemma 2.7.1. *Let $m = d(e, g) = d(e, g^{-1})$. Then*

$$\left\|\lambda_x^n - \lambda_y^n\right\|_1 \leq \frac{2\left|G_{m+n} \setminus G_n\right|}{|G_n|}.$$

Proof. Let $z \in [x]_E = [y]_E$. Then $x = g_1 \cdot z$ and $y = g_2 \cdot z$ for some $g_1, g_2 \in \Gamma$. Suppose that $\lambda_x^n(z) \neq \lambda_y^n(z)$. Without loss of generality, assume $\lambda_x^n(z) \neq 0$. Then $g_1 \cdot F \neq G_n$, but $gg_1 \cdot F \notin G_n$. Then $gg_1 \cdot F \in G_{m+n}$. Thus,

$$\left|\{z \in [x]_E \mid \lambda_x^n(z) \neq 0, \lambda_y^n(z) = 0\}\right| \leq |G_{m+n} \setminus G_n|.$$

The remaining case is symmetric and we have our conclusion as desired.

□

Lemma 2.7.2. *There is some $k \in \mathbb{N}$ such that*

$$|G_{m+n} \setminus G_n| \leq k \cdot m$$

for all $n \in \mathbb{N}$.

Proof. Since F is finite, observe that except for finitely many $g \in \Gamma$, $\gamma \cdot F \cap F = \emptyset$. In addition, the set

$$A = \{g \cdot F \mid g \cdot F \cdot h \neq \emptyset\}$$

is invariant under the choice of $h \in \Gamma$. We also know that $G_{n+1} \setminus G_n$ has at most two disjoint elements. From this, it follows that for any $n \in \mathbb{N}$,

$$|G_{n+1} \setminus G_n| \leq 2\,|A|$$

and if we let $k = 2\,|A|$, then

$$|G_{m+n} \setminus G_n| = \sum_{i=0}^{m-1} |G_{n_i+1} \setminus G_{n+i}| \leq k \cdot m.$$

□

From the above two claims, we have that

$$\left\|\lambda_x^n - \lambda_y^n\right\|_1 \leq \frac{2k \cdot m}{|G_n|}$$

for all $n \in \mathbb{N}$. Since $|G_n| \to \infty$ as $n \to 0$, this implies that

$$\left\|\lambda_x^n - \lambda_y^n\right\|_1 \to 0.$$

□

3. Rigidity

We adapt the arguments of [HK05], and by implication [Ada90], [AL91], [Zim84], to use the above results to give a new proof of an old theorem.

The result as stated below is implicitly due to Nicolas Monod and Yehuda Shalom [MS06], who obtain these kinds of rigidity results in the situation Γ_2 is a non-trivial free product. In turn it follows from the classification of groups with infinitely many ends by Dicks and Dunwoody [DD89], that torsion free groups with infinitely many ends are indeed free products. Our proof is more direct than the argument one obtains in this way.

Theorem 3.1. *(Monod-Shalom) Let Γ_1, Γ_2 be countable torsion free groups and assume Γ_2 has infinitely many ends. Let $\Gamma_1 \times \mathbb{Z}$ and $\Gamma_2 \times \mathbb{Z}$ act freely, ergodically, and in a measure preserving manner on standard Borel probability spaces (X, μ), (X', μ') and assume that the action of $\mathbb{Z}$ on X is itself already ergodic.*

Then there is a homomorphism from Γ_1 to Γ_2 with amenable kernel.

Proof. Assume $\pi : X \to X'$ witnesses orbit equivalence and let

$$\alpha : X \times (\Gamma_1 \times \mathbb{Z}) \to \Gamma_2$$

be the induced cocycle, defined by the requirement that for each $x \in X, \gamma \in \Gamma_1, \ell \in \mathbb{Z}$ there exists $k \in \mathbb{Z}$ with

$$\big(\alpha(x, (\gamma, \ell)), k\big) \cdot \pi(x) = \pi((\gamma, \ell) \cdot x).$$

Apply the Furstenberg-Zimmer lemma as found at say B3.1 of [HK05] to obtain a measurable assignment from X to the probability measures on $E(\Gamma_2, S)^*$,

$$X \to P\big(E(\Gamma_2, S)^*\big),$$

$$x \mapsto \mu_x,$$

which is $\mathbb{Z}$-equivariant:

$$\mu_{(e,\ell)\cdot x} = \alpha(x, (e, \ell)) \cdot \mu_x.$$

Following a standard exhaustion argument (see [Ada90], [HK05] B4.1), we can assume that the cardinality of the support of μ_x is almost everywhere constant and is almost everywhere maximal – that is to say, if

$$X \to P(E(\Gamma_2, S)^*),$$

$$x \mapsto \hat{\mu}_x,$$

is again $\mathbb{Z}$-equivariant, then the cardinality of the support of μ_x is almost everywhere as large as the cardinality of the support of $\hat{\mu}_x$.

Case(i): Almost everywhere we have μ_x supported on one or two points.

Here we can assume that μ_x is a.e. distributed evenly. So in the case it has support of one point, φ, it will be the Dirac measure, δ_φ, on that point; in the case that it has support two points, it will be of the form

$$\mu_x = \frac{1}{2}(\delta_{\varphi_1} + \delta_{\varphi_2}).$$

Lemma 3.1.1. *For $g \in \Gamma_1$, the assignment*

$$x \mapsto \alpha\big(g \cdot x, (g,0)\big)^{-1} \cdot \nu_{(g,0)\cdot x}$$

is again $\mathbb{Z}$-equivariant.

Proof. Let $\nu_x = \alpha\big((g,0) \cdot x, (g,0)\big)^{-1} \cdot \nu_{(g,0)\cdot x}$. Since for any $\ell \in Z$ we have

$$\alpha\big(x,(e,\ell)\big) \cdot \nu_x = \alpha\big(x,(e,\ell)\big) \cdot \alpha\big((g,0)\cdot x,(g,0)\big)^{-1} \cdot \nu_{(g,0)\cdot x},$$

which by the commutativity of the Γ_1 and $\mathbb{Z}$ actions equals

$$\alpha\big((g,\ell)\cdot x,(g,\ell)\big)^{-1}\alpha\big((g,0)\cdot x,(e,\ell)\big) \cdot \nu_{(g,0)\cdot x}$$

$$\alpha\big((g,\ell)\cdot x,(g,\ell)\big)^{-1}\nu_{(g,\ell)\cdot x} = \nu_{(e,\ell)\cdot x}.$$

□

But then we would be able to create a new equivariant assignment of measures with

$$x \mapsto \frac{1}{2}(\mu_x + \alpha(g \cdot x,(g,0))^{-1} \cdot \nu_{(g,0)\cdot x}.$$

The only way that this could fail to have larger support, and hence by giving a contradiction to our choice of μ_x, is if

$$\alpha\big(g \cdot x,(g,0)\big)^{-1} \cdot \nu_{(g,0)\cdot x} = \mu_x$$

almost everywhere.

There was nothing specific about choice of g, and so quantifying over all elements of Γ_1 we in fact obtain

$$x \mapsto \mu_x$$

a $\Gamma_1 \times \mathbb{Z}$ equivariant assignment.

Subcase(ia): There are finitely many ends, $\varphi_1, \varphi_2, \ldots, \varphi_n$, such that for a.e. x, μ_x is supported on $\varphi_1, \ldots, \varphi_n$.

For notational simplicity let us assume the measure μ_x is a.e. supported on a single point. (The support on two points case is entirely similar.) We may fix some non-null $A \subseteq X$ such that for all $x \in A$,

$$\mu_x = \delta_{\varphi_{i_0}},$$

some i_0. But then, using that the stabilizer of a single end has infinite index, Proposition 2.6, we choose $\gamma \in \Gamma_2$ such that $\gamma \cdot \varphi_{i_0} \notin \{\varphi_1, \ldots \varphi_n\}$. Since π is an orbit equivalence we can measurably choose for each $x \in A$ some $f(x) \in X \cap [x]$ with

$$\gamma \cdot \pi(x) = \pi(f(x)).$$

The function $x \mapsto f(x)$ is measure preserving, since it is a morphism included in the equivalence relation, and yet for all $x' \in f[A]$ we have

$$\mu_{x'} = \delta_{\gamma \cdot \varphi_{i_0}},$$

contradicting a.e. every point in X having the associated measure supported on $\varphi_1, \varphi_2, ...\varphi_n$.

Subcase(ib): There does not exist a set of finitely many ends on which μ_x is supported a.e.

Again, let us make the simplifying assumption that μ_x has support a single end a.e; the case where it is supported on two ends a.e. is only notationally more complex. For each x in the corresponding conull set let φ_x be the end on which μ_x is supported. The equivariance assumption on the measures translates across to give

$$\varphi_{(\gamma,\ell)\cdot x} = \alpha(x, (\gamma, \ell)) \cdot \mu_x.$$

Consider X^3 equipped with the measure μ^3. Our case assumptions imply that the set A of triples (x_1, x_2, x_3) for which $\varphi_{x_1}, \varphi_{x_2}, \varphi_{x_3}$ are distinct is non-null in (X^3, μ^3).

Now we are going to seriously use that π is an orbit equivalence. For each $\gamma \in \Gamma_2$ we define a corresponding action on (a conull subset of) X by

$$\gamma \cdot x = \pi^{-1}(\gamma \cdot \pi(x)).$$

The $\Gamma_1 \times \mathbb{Z}$ equivariance gives

$$\gamma \cdot \varphi_x = \varphi_{\gamma \cdot x}$$

for all $\gamma \in \Gamma_2$, a.e. $x \in X$. In particular we obtain that if $(x_1, x_2, x_3) \in A$ then

$$(\gamma \cdot x_1, \gamma \cdot x_2, \gamma \cdot x_3) \in A.$$

Fix $S \subseteq \Gamma_2$ some finite connected set such that for a non-null collection of $(x_1, x_2, x_3) \in A$ we have S separating $\varphi_{x_1}, \varphi_{x_2}, \varphi_{x_3}$ – i.e.

$$\varphi_{x_1}(S), \varphi_{x_2}(S), \varphi_{x_3}(S)$$

all distinct. We then let B be the set of (x_1, x_2, x_3) for which some Γ_2 translate of S separates $\varphi_{x_1}, \varphi_{x_2}, \varphi_{x_3}$. This subset of A will be invariant under our induced action of Γ_2.

Now we can proceed to the analogue of the Russ Lyons trick (see for instance C2.3 [HK05]). For each $\vec{x} = (x_1, x_2, x_3) \in B$ we let $\mathcal{F}_{\vec{x}}$ be the collection of $\gamma \cdot S$ which separate the three associated ends; the equivariance assumptions yield that

$$\mathcal{F}_{\gamma \cdot \vec{x}} = \gamma \cdot \mathcal{F}_{\vec{x}} =_{df} \{\gamma \cdot S' : S' \in \mathcal{F}_{\vec{x}}\}.$$

By Proposition 2.4 we have each $\mathcal{F}_{\vec{x}}$ finite, and in fact there will be a uniform upper bound. We then let ν_x be a finite measure on Γ_2 which has

$$\nu_{\vec{x}}(\sigma) = 1$$

if $\sigma \in S'$ for some $S' \in \mathcal{F}_{\vec{x}}$, and

$$\nu_{\vec{x}}(\sigma) = 0$$

otherwise. We can then define a measure ν on Γ_2 by $\nu = \int \nu_{\vec{x}} d\mu^3(\vec{x})$ – that is to say,

$$\nu(\{\sigma\}) = \mu^3(\{\vec{x} : \exists S' \in \mathcal{F}_{\vec{x}}(\sigma \in S')\}).$$

Lemma 3.1.2. *ν is Γ_2 invariant.*

Proof. Fix $\gamma \in \Gamma_2$.

$$\begin{aligned}\nu(\{\gamma^{-1}\sigma)\}) &= \mu^3\big(\{\vec{x} : \exists S' \in \mathcal{F}_{\vec{x}}(\gamma^{-1}\sigma \in S'\}\big)\\ &= \mu^3\big(\{\vec{x} : \exists S' \in \gamma \cdot \mathcal{F}_{\vec{x}}(\sigma \in S'\}\big),\end{aligned}$$

which by equivariance equals

$$\mu^3\big(\{\vec{x} : \exists S' \in \mathcal{F}_{\gamma\cdot\vec{x}}(\sigma \in S')\}\big) = \mu^3\big(\gamma^{-1} \cdot \{\vec{x} : \exists S' \in \mathcal{F}_{\vec{x}}(\sigma \in S')\}\big),$$

but since the induced actions of Γ_2 on X and then X^3 are measure preserving, this in turn evaporates away to give

$$\mu^3\big(\{\vec{x} : \exists S' \in \mathcal{F}_{\vec{x}}(\sigma \in S')\}\big) = \nu(\{\sigma\}),$$

as required. □

But now we have the radically ridiculous situation of ν providing an invariant finite measure on an infinite group.

Case(ii): μ_x is not supported on one or two points a.e.

Again we can find some single finite connected $S \subseteq \Gamma_2$ such that for a μ positive collection of $x \in X$ there is a $(\mu_x)^3$ positive collection of $(\varphi_1, \varphi_2, \varphi_3)$ for which S separates $\varphi_1, \varphi_2, \varphi_3$. By ergodicity we obtain for μ-a.e. x there is some translate $\gamma \cdot S$ of S for which there is a $(\mu_x)^3$ positive set on which $\gamma \cdot S$ separates the three ends.

For each x in this conull set we let $r_x > 0$ be the largest positive r for which there exists some $\gamma \cdot S$ with

$$(\mu_x)^3\big(\{(\varphi_1, \varphi_2, \varphi_3) \mid \varphi_1(\gamma \cdot S), \varphi_2(\gamma \cdot S), \varphi_3(\gamma \cdot S) \text{ all distinct}\}\big) = r.$$

Again by ergodicity, there is a single r_0 such that $r_x = r_0$ a.e. For each x in this conull set we let $\mathcal{H}_x$ be the collection of all $\gamma \cdot S$ such that

$$(\mu_x)^3\big(\{(\varphi_1, \varphi_2, \varphi_3) \mid \varphi_1(\gamma \cdot S), \varphi_2(\gamma \cdot S), \varphi_3(\gamma \cdot S) \text{ all distinct}\}\big) = r_0.$$

Appealing to Proposition 2.4 and recalling that the measure μ_x is finite we conclude that $\mathcal{H}_x$ will be a finite set of finite subsets of Γ_2.

We let Γ_2 act on the finite collections of finite subsets of Γ_2 in the usual way,

$$\gamma \cdot \mathcal{H} = \{\gamma \cdot T : T \in \mathcal{H}\}.$$

Since Γ_2 is torsion free, this action will be free. $\mathbb{Z}$-equivariance gives that for all $\ell \in \mathbb{Z}$ and a.e. $x \in X$

$$\mathcal{H}_{(e,\ell)\cdot x} = \alpha(x,(e,\ell)) \cdot \mathcal{H}_x.$$

Then by ergodicity of the $\mathbb{Z}$ action on X we can find some single $\mathcal{H}_0$ such that a.e. $\mathcal{H}_x$ is a translate of $\mathcal{H}_0$. For a.e. x we can then let $g_x \in \Gamma_2$ be the unique element with

$$g_x \cdot \mathcal{H}_x = \mathcal{H}_0.$$

We can then define a new reduction

$$\pi^* : X \to Y$$

$$x \mapsto g_x \cdot \pi(x),$$

with associated cocyle

$$\alpha^* : X \times (\Gamma_1 \times \mathbb{Z}) \to \Gamma_2,$$

$$\big(x,(\gamma,\ell)\big) \mapsto g_{(\gamma,\ell)\cdot x}\alpha(x,(\gamma,\ell))g_x^{-1}.$$

It follows from the structure of the definitions that

$$\alpha^*(x,(e,\ell)) \cdot \mathcal{H}_0 = g_{(e,\ell)\cdot x}\alpha\big(x,(e,\ell)\big)g_x^{-1}\mathcal{H}_0 =$$

$$g_{(e,\ell)\cdot x}\alpha\big(x,(e,\ell)\big)\mathcal{H}_x = g_{(e,\ell)\cdot x}\mathcal{H}_{(e,\ell)\cdot x} = \mathcal{H}_0,$$

and thus $\alpha^*\big(x,(e,\ell)\big)$ is equal to the identity in Γ_2 a.e.

But now we obtain a homomorphism from Γ_2 in the usual way. For each $\gamma \in \Gamma_1$ we define a function

$$f_\gamma : X \to \Gamma_2$$

$$x \mapsto \alpha^*(x,(\gamma,0)).$$

The function is invariant under the action of $\mathbb{Z}$, and hence, by ergodicity, constant a.e. So we can let $\rho(\gamma)$ be that a.e. constant value of $f_\gamma(x)$. Since

$$f_{\gamma_1}(\gamma_2 \cdot x)f_{\gamma_2}(x) = f_{\gamma_1\gamma_2}(x)$$

we obtain that ρ is a homomorphism. Let Δ be the kernel. Our journey home will be over once we verify Δ is a amenable.

But for any $h \in \Delta$ we have a.e. $x \in X$ with

$$\alpha^*(x,(h,0)) = e,$$

and thus there must be some $\ell \in \mathbb{Z}$ such that

$$\pi^*((h,0)\cdot x) = (e,\ell)\cdot \pi^*(x).$$

Thus we have a Borel reduction of the equivalence relation induced by Δ to the equivalence relation induced by the action of $\{e_{\Gamma_2}\} \times \mathbb{Z}$, which implies the former group is amenable. □

4. Selection of an end

The connection between countable Borel equivalence relations and the ability to select a given number of ends from each component of a particular graphing has been previously explored in the measure theoretic setting by Adams [Ada90], Paulin [Pau99] and Jackson-Kechris-Louveau [JKL02]. In the descriptive set theoretic setting, Miller [Mil] recently showed that if a Borel equivalence relation admits a graphing whose number of ends is finite and is at least two, then the equivalence relation must be hyperfinite. This is also the case for equivalence relations that admit graphings for which one can select, in a Borel fashion, from each component a number of ends that is finite and is at least two. Unfortunately, every countable Borel equivalence relation with infinite classes admits a locally finite single-ended graphing. Even assuming infinitely many ends, the ability to select two does not place any bound on the complexity of the equivalence relation.

Example Let $\Gamma = (\mathbb{F}_2 \times \mathbb{Z}) * \mathbb{F}_2$ and as in the previous aside in Section 2.3, take the canonical generating set, S. We let X be the standard Borel space of all functions

$$f : \Gamma \to \{0, 1\}$$

where f assumes non-zero values on *exactly one* of the right cosets of $\mathbb{F}_2 \times \mathbb{Z}$. Set $f_1 \mathcal{R} f_2$ if there is some

$$\sigma \in S^{\pm 1}$$

such that $\sigma \cdot f_1 = f_2$, where we define the usual shift action by

$$\sigma \cdot f(\tau) = f(\sigma^{-1}\tau).$$

Given $f \in X$, we obtain different ends of $[f]_\Gamma$ by for instance considering the various right cosets of $\mathbb{F}_2 \times \mathbb{Z}$. For each right coset

$$(\mathbb{F}_2 \times \mathbb{Z})g$$

we choose a ray inside $\{\sigma g \cdot f : \sigma \in \mathbb{F}_2 \times \mathbb{Z}\}$. Then we obtain a resulting end φ with

$$\varphi(F)$$

the connected component of $C(\Gamma, S) \setminus F$ containing all but finitely many points in the ray. Moreover, we do have the ability to select an end inside each equivalence class: Namely choose the coset of $\mathbb{F}_2 \times \mathbb{Z}$ on which f achieves some non-zero values. Finally, he equivalence relation is not amenable, in fact it has no reasonable bound on its complexity. We can Borel reduce $E_{\mathbb{F}_2 \times \mathbb{Z}}^{2^{\mathbb{F}_2 \times \mathbb{Z}}}$ to E_Γ^X.

The example above, however, did not come equipped with a Borel probability measure, and this is the critical issue. If we assume infinitely many ends *and* the existence of an invariant Borel probability measure, then there is no way of choosing an end for each class.

Theorem 4.1. *Suppose that E is an ergodic, measure preserving equivalence relation on standard probability space (X, μ) that admits a graphing $\mathcal{R}$ with infinitely many ends. Then there is no E-invariant Borel assignment*

$$x \mapsto \varphi_x \in \partial(\mathcal{R}|[x]_E).$$

Proof. Assume, for the purpose of contradiction, that there is such an assignment. Let $[X]^{<\infty}$ denote the standard Borel space of finite subsets of X and let

$$[E]^{<\infty} = \{S \in [X]^{<\infty} \mid \forall x, y \in S \quad xEy\} \subseteq [X]^{<\infty}$$

denote the set of pairwise E-related finite subsets of X. First, let $x \mapsto \{F_n^x\}_{n\in\mathbb{N}}$ be a Borel assignment of enumerations of the finite connected subsets of $[x]_E$. Indeed, the set

$$C^{<\infty} = \{S \in [X]^{<\infty} \mid S \text{ is connected }\}$$

is Borel and so is $C^{<\infty} \cap [E]^{<\infty}$. Given a Borel ordering $<_X$ on X, one can use a lexicographical ordering on $C^{<\infty} \cap [E]^{<\infty}$ to obtain the enumeration.

Then for each $x \in X$, there is some $F \in \{F_n^x\}_{n\in\mathbb{N}}$ such that $\varphi(F) \neq \varphi_x(F)$ for some $\varphi \in \partial(\mathcal{R}|[x]_E)$. In particular, $[x]_E \setminus \varphi_x(F)$ is infinite. Then if we let N_x be the size of the smallest such F in the enumeration, the assignment $x \mapsto N_x$ is E-invariant and, by ergodicity of E, is constant almost everywhere. Thus, let N be such $N = N_x$ for almost every $x \in X$.

Lemma 7.3 of Kechris-Miller [KM04] guarantees the existence of a finite Borel subequivalence relation E_N on a Borel subset $Y \subseteq X$ such that

(1) $\forall x \in Y, [x]_{E_N}$ is connected in $\mathcal{R}|[x]_E$;
(2) Y is an almost everywhere complete section for E.

Then if we let

$$\mathcal{F}_x = \left\{[y]_{E_N} \mid yEx\right\} \subseteq \left[[x]_E\right]^N,$$

then $x \mapsto \mathcal{F}_x$ is an E-invariant Borel assignment to disjoint connected subsets of $[x]_E$ such that for all $F \in \mathcal{F}_x$, $[x]_E \setminus \varphi_x(F)$ is infinite. Observe that $\mathcal{F}_x$ is infinite almost everywhere. Otherwise, by ergodicity of E, the assignment $x \mapsto |\mathcal{F}_x| < \infty$ is constant almost everywhere and we have a Borel E-invariant way of selecting a finite set of elements from each E-equivalence class, contradicting the non-amenability of E.

We say that $F \in \mathcal{F}_x$ is *bad* if

$$\forall F' \in \mathcal{F}_x \quad F' \subseteq \varphi_x(F) \implies F \subseteq \varphi_x(F').$$

Lemma 4.1.1. *For almost every $x \in X$, there are only finitely many bad $F \in \mathcal{F}_x$ and, thus, we may assume that none are bad.*

Proof. First, note that for two distinct bad $F_1, F_2 \in \mathcal{F}_x$, we must have $\varphi_x(F_1) \cup \varphi_x(F_2) = [x]_E$. Indeed, if we let $y \notin \varphi_x(F_1) \cup \varphi_x(F_2)$, then let us consider a path from y to $F_1 \cup F_2$. Without loss of generality, assume the path reaches F_1 first. Then since F_1 and F_2 are disjoint, we must have that $F_1 \subseteq \varphi_x(F_2)$ or $F_2 \subseteq \varphi_x(F_1)$. By badness, both these inclusions hold and, as a result, $y \in \varphi_x(F_2)$, leading to a contradiction.

Now let

$$\mathcal{A}_x = \{F \in \mathcal{F}_x \mid F \text{ is bad}\}.$$

Then $x \mapsto \mathcal{A}_x$ is also an E-invariant Borel assignment so that $\mathcal{A}_x$ is infinite.

$$Y = \{y \in X \mid \exists F \in \mathcal{A}_x, \quad y \notin \varphi_x(F)\}.$$

Then $\mu(Y) > 0$ since $\mu([Y]_E) > 0$. We now define an equivalence relation F on Y by

$$y_1 F y_2 \iff y_1 E y_2 \text{ and } \exists F \in \mathcal{A}_{y_1} = A_{y_2} \quad y_1, y_2 \notin \varphi_x(F).$$

Then this is a measure preserving equivalence relation with infinite classes. However, if we then for $y \in Y$, let F_y be the one element in $\mathcal{A}_y$ such that $y \notin F_y$, then this assignment is F-invariant and F must be smooth, leading to a contradiction. □

Then we may then obtain a Borel assignment

$$x \mapsto \{F_n^x\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_x$$

such that

(1) $F_{n+1}^x \subseteq \varphi_x(F_n^x)$ and $F_n^x \not\subseteq \varphi_x(F_{n+1}^x)$;
(2) F_{n+1}^x so that the distance from F_{n+1}^x to F_n^x is minimal;
(3) if $F_i^x = F_j^y$, then $F_{i+1}^x = F_{j+1}^y$.

Lemma 4.1.2. *Suppose that $F_1, F_2, F_3 \in \mathcal{F}_x$ and the following hold:*

(1) *$F_2 \subseteq \varphi_x(F_1)$, $F_3 \subseteq \varphi_x(F_2)$;*
(2) *$F_1 \not\subseteq \varphi_x(F_2)$, $F_3 \not\subseteq \varphi_x(F_3)$.*

Then $F_3 \subseteq \varphi_x(F_1)$ and $F_1 \not\subseteq \varphi_x(F_3)$. In particular, in the above sequences, if $j > i$, then

$$F_j^x \subseteq \varphi_x(F_i^x), \quad F_i^x \not\subseteq \varphi_x(F_j^x).$$

Proof. There is a path inside $\varphi_x(F_2)$ from F_2 to F_3 such that $p_2 \cap F_1 = \emptyset$ since $F_1 \not\subseteq \varphi_x(F_2)$. Then since $F_2 \subseteq \varphi_x(F_1)$, $F_3 \subseteq \varphi_x(F_1)$.

If $F_1 \subseteq \varphi_x(F_3)$, then there is path inside $\varphi_x(F_3)$ from F_3 to F_1 avoiding F_2 since $F_2 \not\subseteq \varphi_x(F_3)$. Now, since $F_3 \subseteq \varphi_x(F_2)$, this implies that $F_1 \subseteq \varphi_x(F_2)$ also and we obtain a contradiction. □

Lemma 4.1.3. *For almost every $x \in X$, given $T \in \mathcal{F}_x$, there exists some $n \in \mathbb{N}$ such that for all $i \geq n$, $F_i^x \subseteq \varphi_x(T)$.*

Proof. Suppose that for some $l \in \mathbb{N}$, $F_l^x \not\subseteq \varphi_x(T)$ since otherwise we are done. Let p be a path from F_l^x to T and let $n \in \mathbb{N}$ be such that $n > l$ and for all $i \geq n$, F_i^x avoids p. Then for $i \geq n$,

$$T \subseteq \varphi_x(F_i^x) \text{ or } F_i^x \subseteq \varphi_x(T).$$

The former cannot happen since it would imply that $F_l^x \subseteq \varphi_x(F_i^x)$ and contradict the previous lemma so we are done. □

Lemma 4.1.4. *If xEy, then there exist $i, j \in \mathbb{N}$ such that $F_i^x = F_j^y$.*

Proof. Suppose that this is not the case. First, let $n \in \mathbb{N}$ be obtained by the previous lemma such that

$$\forall i \geq n \quad F_i^y \subseteq \phi_y(F_0^x) = \phi_x(F_0^x)$$

and the distance from F_0^x to F_n^y is the minimum possible given the above constraint. It must then be that

$$F_0^x \not\subseteq \phi_x(F_{n+1}^y).$$

Indeed, there is then a path from F_{n+1}^y to F_0^x inside $\phi_x(F_{n+1}^y)$ and a path from F_0^x to F_n^y avoiding F_{n+1}^y, leading to the conclusion that $F_n^y \subseteq \phi_x(F_{n+1}^y)$.

As a result, by the previous lemma again, we may let $k \in \mathbb{N}$ be least such that

$$F_k^x \subseteq \phi_x(F_{n+1}^y).$$

Let p_0 be a path from F_{k-1}^x to F_k^x of minimal length inside $\phi_x(F_{k-1}^x)$. Observe that p_0 avoids F_{n+1}^y. Indeed, otherwise $F_{n+1}^y \subseteq \phi_x(F_{k-1})$ and the distance from F_{n+1}^y to F_{k-1}^x is smaller than the distance from F_k^x to F_{k-1}^x. Since

$$F_{k-1}^x \not\subseteq \phi_x(F_{n+1}^y)$$

by our choice of k, this would entail that $F_k^x = F_{n+1}^y$.

Furthermore, let p_1 be a path from F_{n+1}^y to F_k^x inside $\phi_x(F_{n+1}^y)$. Obtain the path p_2 from p_0 by reversing the order of the path p_0. Now $p_1 p_2$ gives a path starting in $\phi_x(F_{n+1}^y)$ and ending in F_{k-1}^x avoiding F_{n+1}^y. We may thus conclude that $F_{k-1}^x \subseteq \phi_x(F_{n+1}^y)$, leading to a contradiction to our choice of k. □

As a result, we have

$$xEy \implies \exists i, j \in \mathbb{N} \quad \forall n \in \mathbb{N} \quad F_{i+n}^x = F_{j+n}^y.$$

Let $f_n^x \in l^1([x]_E)$ be defined by

$$f_n^x = \frac{1}{N \cdot n} \sum_{i=0}^{n-1} \mathbf{1}_{F_i^x}.$$

Then each $f_n^x \in l^1([x]_E)$, $\|f_n^x\|_{l^1([x]_E)} = 1$ and

$$\forall xEy \quad \|f_n^x - f_n^y\|_{l^1([x]_E)} \to 0.$$

This witnesses that E is amenable, contradicting that E has infinitely many ends. □

5. Equivalence relations and percolation

5.1. **Amenable subequivalence relations.** Let (X, μ) be a standard probability space and E a Borel equivalence relation on X. Following Kechris-Miller [KM04], we say a subequivalence relation $F \subseteq E$ is *normal* if there is a countable group of Borel automorphisms $G = \{g_i\}_{i \in \mathbb{N}}$ that generates E and such that each g_i preserves F in the sense that

$$xFy \iff g_i(x) F g_i(y) \quad \forall i \in \mathbb{N} \quad \forall x, y \in X.$$

As an example, suppose that Δ is a normal subgroup of a countable group Γ and $\Gamma \curvearrowright (X,\mu)$. Let E_Γ, E_Δ be the orbit equivalence relations obtained from the action of Γ and the action of Γ restricted to Δ, respectively. E_Δ is then clearly normal in E_Γ.

For a compact metric space K, we denote by $\mathcal{M}(K)$ the compact space of measures on K (see [Kec95]). Furthermore, we let

$$\mathcal{M}_{\leq 2}(K) = \{\nu \in \mathcal{M}(K) \mid \quad |\mathrm{supp}(\nu)| \leq 2\}$$
$$\mathcal{M}_3(K) = \mathcal{M}(K) \setminus \mathcal{M}_{\leq 2}(K).$$

As mentioned previously, let $\partial\mathcal{G}$ be the compact space of ends of a countable graph $\mathcal{G}$.

Proposition 5.1. *Suppose that E is a measure preserving equivalence relation on standard probability space X and $F \subseteq E$ is an ergodic subequivalence relation that admits a locally finite graphing $\mathcal{R}$ such that for almost every $x \in X$, $\mathcal{R}|[x]_F$ has infinitely many ends. Let $F_0 \subseteq E$ be a normal subequivalence relation. Then $F \cap F_0$ is non-amenable or smooth almost everywhere.*

Proof. Let $\{g_i\}_{i\in\mathbb{N}}$ be a countable enumeration of automorphism group elements witnessing that F_0 is normal in E. For the purpose of contradiction, suppose that $F \cap F_0$ is actually amenable. Following the methods of Hjorth-Kechris [HK05] and using the Furstenberg-Zimmer lemma (see Chapter 4 of [Zim84]), consider a μ-measurable map

$$x \mapsto \nu_x \in \mathcal{M}(\partial(\mathcal{R}|[x]_F))$$

that is $F_0 \cap F$-invariant almost everywhere, i.e.,

$$x(F_0 \cap F)y \implies \nu_x = \nu_y$$

for almost every $x, y \in X$.

One situation is that $\nu_x \in \mathcal{M}_{\leq 2}\big(\partial(\mathcal{R}|[x]_F)\big)$ for a subset of X of positive measure. This set is, of course, $F \cap F_0$-invariant. We may assume that the map $x \mapsto \nu_x$ is maximum and claim that it is F-invariant as well.

Fix $i \in \mathbb{N}$. Let

$$A_i = \{x \in X \mid g_i(x) F x\}.$$

and consider the measure $\nu_x^{g_i} = \nu_{g_i(x)}$. We claim that the assignment $x \mapsto \nu_x^{g_i}$ is also $F \cap F_0$-invariant on $[A_i]_{F_0 \cap F}$. Suppose that xF_0y. Then by normality of F_0 in E, we have

$$g_i(x) F_0 g_i(y).$$

Thus,

$$\nu_{g_i(x)} = \nu_{g_i(y)} \text{ and } \nu_x^{g_i} = \nu_y^{g_i}.$$

Consequently, for almost every $x \in [A_i]_{F_0\cap F}$, by maximality of ν_x, we must have

$$\mathrm{supp}(\nu_x^{g_i}) \subseteq \mathrm{supp}(\nu_x).$$

So then,

$$\nu_x^{g_i} = \nu_x$$

for almost every $x \in [A_i]_{F_0 \cap F}$. Since this is the case for every $i \in \mathbb{N}$, the assignment $x \mapsto \nu_x$ is an F-invariant assignment.

Now, by ergodicity of F, either ν_x concentrates on two ends in $\partial(\mathcal{R}|[x]_F)$ for almost every $x \in X$ or on one end for almost every $x \in X$. Theorem 4.1 precludes the latter situation. Theorem D of Miller [Mil] ensures that every countable Borel equivalence relation with a graphing for which it is possible to select two ends from every component must be hyperfinite. As a result, F must be amenable, leading to a contradiction.

It remains that $\nu_x \in \mathcal{M}_3\big(\partial(\mathcal{R}|[x]_F)\big)$ for almost every $x \in X$. Since ν_x is supported on at least three ends for almost every $x \in X$, the set

$$\big[\partial(\mathcal{R}|[x]_F)\big]^3_0 \subseteq \big[\partial(\mathcal{R}|[x]_F)\big]^3$$

of triples of distinct ends has positive $(\nu_x)^3$- measure for almost every $x \in X$. For

$$(\phi_1, \phi_2, \phi_3) \in \big[\partial(\mathcal{R}|[x]_F)\big]^3_0,$$

let $S(\phi_1, \phi_2, \phi_3)$ be the finite connected subsets of $[x]_F$ of least cardinality that separate ϕ_1, ϕ_2, ϕ_3. As mentioned earlier, if $F_1, F_2 \in S(\phi_1, \phi_2, \phi_3)$, then $F_1 \cap F_2 \neq \emptyset$. Since $\mathcal{R}$ is locally finite, this implies that each $S(\phi_1, \phi_2, \phi_3)$ is finite. Let

$$F(\phi_1, \phi_2, \phi_3) = \bigcup S(\phi_1, \phi_2, \phi_3).$$

Define the functions $h_x \in l^1([x]_{F \cap F_0})$ by

$$h_x(y) = \int_{\big[\partial(\mathcal{R}|[x]_F)\big]^3_0} \frac{1}{F(\phi_1, \phi_2, \phi_3)} 1_{F(\phi_1,\phi_2,\phi_3)}(y) \, \mathrm{d}\nu_x(\phi_1, \phi_2, \phi_3).$$

Now, if $x(F \cap F_0)y$, then $h_x = h_y$. As a result, we may let

$$a(x) = \max\{h_x(y) \mid y(F \cap F_0)x\}$$

and then let

$$A = \{x \in X \mid h_x(x) = a(x)\}.$$

Then A is an almost everywhere Borel complete section for $F \cap F_0$ such that only intersects every $F \cap F_0$ class in finitely many points. This establishes that $F \cap F_0$ is smooth.

□

Corollary 5.2. *If E is a measure preserving equivalence relation that admits a locally finite graphing with infinitely many ends, then E does not admit a normal ergodic amenable subequivalence relation.*

Recall that an equivalence relation E is *amenable* if there exists a sequence $\lambda_n \colon E \to \mathbb{R}$ of non-negative Borel functions such that

(1) $\lambda^n_x \in l^1([x]_E)$ where $\lambda^n_x(y) = \lambda^n(x, y)$ for xEy;
(2) $\|\lambda^n_x\|_1 = 1$;

(3) there is a Borel E-invariant set $A \subseteq X$ with $\mu(A) = 1$ such that
$$\left\|\lambda^n_x - \lambda^n_y\right\|_1 \to 0$$
as $n \to \infty$ for all xEy with $x, y \in A$.

Following Jackson-Kechris-Louveau [JKL02], E is *2-amenable* if there exists a doubly-indexed sequence $\{\lambda_{n,m} \colon E \to \mathbb{R}\}_{n,m\in\mathbb{N}}$ of non-negative Borel functions such that conditions (1) and (2) hold as above and also there is a Borel E-invariant set $A \subseteq X$ with $\mu(A) = 1$ such that for all $x, y \in A$ such that xEy and $\epsilon > 0$, there is some $M \in \mathbb{N}$ such that for all $m \geq M$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$,
$$\left\|\lambda^{n,m}_x - \lambda^{n,m}_y\right\|_{l^1([x]_E)} < \epsilon.$$
Proposition 2.13 of Jackson-Kechris-Louveau shows that a Frechet-amenable equivalence relation (this includes 2-amenable equivalence relations) is μ-amenable, but this uses the Continuum Hypothesis. By Appendix 2 of Adams-Lyons [AL91], the statement "E on (X, μ) is amenable" is a projective statement and is true in ZFC without the assumption of CH.

Definition Suppose that $E \subseteq F$ are countable equivalence relations on standard probability space (X, μ). We say that F *is given by a* $\mathbb{Z}$*-action over* E if there is an action $\mathbb{Z} \curvearrowright X/E$ such that for a.e. $x \in X$, $[x]_F = \bigcup(\mathbb{Z} \cdot [x]_E)$ and there is a Borel function
$$\pi : X \times \mathbb{Z} \to X$$
such that $[\pi(x, \ell)]_E = \ell \cdot [x]_E$ for all $x \in X$, $\ell \in \mathbb{Z}$.

Proposition 5.3. *If E is amenable and F is given by a $\mathbb{Z}$-action over E, then F is also amenable.*

Proof. Let $\lambda^n \colon E \to \mathbb{R}$ be a sequence of functions witnessing the amenability of $E_\Gamma \cap F$. Given $x \in X$, let $F^k_x = k \cdot [x]_E$ (this is the k-th equivalence class from x). We will need the following lemma:

Lemma 5.3.1. *There exists a Borel sequence of functions $\{\varphi_k\}_{k\in\mathbb{Z}} \colon X \to X$ such that $\varphi_k(x) \in F^k_x$.*

Proof. Let $G \curvearrowright (X, \mu)$ be such that $E_G = F$. Then let $\{g_n\}_{n\in\mathbb{N}}$ be an enumeration of G. Then define $f_k \colon X \to \mathbb{N}$ by letting $g_k(x)$ be the least $n \in \mathbb{N}$ such that $g_n(x) \in k \cdot [x]_E$. Then it is routine to check that each g_k is Borel and we may let $\varphi_k(x) = g_{f(x)} \cdot x$. □

We then define functions $\lambda^{n,m}_x \in l^1([x]_F)$ as follows:
$$\lambda^{n,m}_x = \sum_{|l|\leq m} \frac{\lambda^n_{\varphi_l(x)}}{2m+1}.$$

It remains to establish that the doubly-indexed sequence $\{\lambda^{n,m}\}_{n,m\in\mathbb{N}}$ witnesses the 2-amenability of F. For this purpose, let xFy. Then there is some

$k \in \mathbb{Z}$ such that $k \cdot [x]_E = [y]_E$. Without loss of generality, we may assume that k is non-negative. We then let M be such that $k/M < \epsilon$. Then let $m > M$.

Note that for every $l \in \mathbb{Z}$, $F^l_y = F^{l+k}_x$ and, as a result, $\varphi_l(y) E \varphi_{l+k}(x)$. By our choice of the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, for every $l \in \mathbb{Z}$, there is some $N_l \in \mathbb{N}$ such that for $n \geq N_l$,

$$\left\| \lambda^n_{\varphi_l(y)} - \lambda^n_{\varphi_{l+k}(x)} \right\|_{l^1([\varphi_l(y)]_E)} < \epsilon.$$

Let $N = \max\{N_l \mid -m \leq l \leq m-k\}$ and let $n \geq N$. Then

$$\begin{aligned} &\left\| \lambda^{n,m}_x - \lambda^{n,m}_y \right\|_{l^1([x]_F)} \\ &= \left\| \sum_{|l|\leq m} \frac{\lambda^n_{\varphi_l(x)}}{2m+1} - \sum_{|l|\leq m} \frac{\lambda^n_{\varphi_l(y)}}{2m+1} \right\|_{l^1([x]_F)} \\ &\leq \frac{1}{2m+1} \left\| \sum_{l=-m+k}^{m} \lambda^n_{\varphi_l(x)} - \sum_{l=-m}^{m-k} \lambda^n_{\varphi_l(y)} \right\|_{l^1([x]_F)} && (5.1) \\ &+ \frac{1}{2m+1} \left\| \sum_{l=-m}^{-m+k-1} \lambda^n_{\varphi_l(x)} - \sum_{l=m-k+1}^{m} \lambda^n_{\varphi_l(y)} \right\|_{l^1([x]_F)} && (5.2) \end{aligned}$$

We now calculate (5.1) and (5.2) separately. Note that for $n \in \mathbb{N}$ and $l \in \mathbb{Z}$, $\lambda^n_{\varphi_l(y)}$ concentrates on $[\varphi_l(y)]_E$. Thus, for (5.1), we have

$$\begin{aligned} &\frac{1}{2m+1} \left\| \sum_{l=-m+k}^{m} \lambda^n_{\varphi_l(x)} - \sum_{l=-m}^{m-k} \lambda^n_{\varphi_l(y)} \right\|_{l^1([x]_F)} \\ &= \frac{1}{2m+1} \left\| \sum_{l=-m}^{m-k} \lambda^n_{\varphi_{l+k}(x)} - \sum_{l=-m}^{m-k} \lambda^n_{\varphi_l(y)} \right\|_{l^1([x]_F)} \\ &\leq \frac{1}{2m+1} \sum_{l=-m}^{m-k} \left\| \lambda^n_{\varphi_{l+k}(x)} - \lambda^n_{\varphi_l(y)} \right\|_{l^1([\varphi_l(y)]_E)} < \frac{(2m+1)\epsilon}{2m+1} = \epsilon. \end{aligned}$$

For (5.2), we have

$$\frac{1}{2m+1}\left\|\sum_{l=-m}^{-m+k-1}\lambda^n_{\varphi_l(x)}-\sum_{l=m-k+1}^{m}\lambda^n_{\varphi_l(y)}\right\|_{l^1([x]_F)}$$
$$\leq \frac{1}{2m+1}\sum_{l=-m}^{-m+k-1}\left\|\lambda^n_{\varphi_l(x)}\right\|_{l^1([\varphi_l(x)])}+\frac{1}{2m+1}\sum_{l=m-k+1}^{m}\left\|\lambda^n_{\varphi_l(y)}\right\|_{l^1([\varphi_l(y)])}$$
$$\leq \frac{k}{2m+1}+\frac{k}{2m+1}<\epsilon+\epsilon=2\epsilon.$$

From the above calculations,

$$\left\|\lambda^{n,m}_x-\lambda^{n,m}_y\right\|_{l^1([x]_F)}<3\epsilon.$$

□

5.2. **Percolation on Cayley graphs.** It turns out that the analysis of equivalence relations admitting graphings with infinitely many ends has some applications to percolation. We recommend the book by Lyons and Peres [LP] for a thorough introduction to the subject.

Let us consider a locally finite countable graph $\mathcal{G}$ - in fact, for our purposes, we will just consider Cayley graphs of finitely generated groups. If we let E be the set of edges in $\mathcal{G}$, then every element in the space $X=\{0,1\}^E$ has a natural identification with a subgraph of $\mathcal{G}$.

A *bond percolation* is a probability measure on the space X. In particular, the *Bernoulli bond percolation* is obtained by equipping $\{0,1\}$ with the Bernoulli(p) measure and X with the product measure for some $p\in[0,1]$. This can be thought of as independently removing edges in $\mathcal{G}$ with probability $1-p$. We will denote the Bernoulli bond percolation with parameter p by μ_p.

There is a very prominent connection between percolation on Cayley graphs of countable groups and countable equivalence relations. Γ acts on E by right translation and this induces a shift action on $\{0,1\}^E$. This gives rise to the *full equivalence relation* E_Γ given by the orbit equivalence relation associated to the action $\Gamma\curvearrowright\{0,1\}^E$. This is clearly ergodic since it is even mixing. Furthermore, the *cluster equivalence relation* E^{cl}_Γ (see Gaboriau [Gab05]) is formed by letting $xE^{cl}_\Gamma y$ if and only if there is some $\gamma\in\Gamma$ such that $\gamma^{-1}\cdot x=y$ and γ is in the connected component of the identity in x.

The number of infinite clusters in x will actually be constant for μ_p-almost every $x\in X$ and there will be 0, 1 or infinitely many infinite clusters.

Associated with any Cayley graph are the constants

$$p_c(\mathcal{G})=\inf\{p\mid\mu_p-\text{a.e. } x\in X \text{ has an infinite cluster}\}$$
$$p_u(\mathcal{G})=\inf\{p\mid\mu_p-\text{a.e. } x\in X \text{ has infinitely many infinite clusters}\},$$

Häggström-Peres [HP99] showed that for a given graph $\mathcal{G}$, there are three phases in $[0,1]$. In particular, there are no infinite clusters μ_p-a.e. for $p<p_c(\mathcal{G})$,

infinitely many infinite clusters μ_p-a.e. for $p_c(\mathcal{G}) < p < p_u(\mathcal{G})$ and exactly one infinite cluster μ_p-a.e. for $p > p_u(\mathcal{G})$. Graphs for which $p_c(\mathcal{G}) < p_u(\mathcal{G})$ are said to have a *non-unique percolation phase*. Pak, Smirnova-Nagnibeda [PSN00] showed that every finitely generated non-amenable group admits a Cayley graph with a non-unique percolation phase. Lyons and Schramm [LS99] showed that for $p > p_c(\mathcal{G})$, the infinite clusters are indistinguishable and the cluster equivalence relation is ergodic when restricted to the μ_p-positive measure set U of elements in $\{0,1\}^E$ with an infinite cluster at the identity.

An *L-graphing* of an equivalence relation E (see Kechris-Miller [KM04]) is a family $\Phi = \{\phi_i\}_{i\in I}$ of partial automorphisms where $\phi_i(x)Ex$ for every $i \in I$ and $x \in X$ such that if xEy, then there is a word w consisting of elements of Φ such that $w(x) = y$. To every L-graphing corresponds a graphing $\mathcal{R}$ where

$$(x, y) \in \mathcal{R} \iff \exists\phi \in \Phi \quad \phi(x) = y \text{ or } \phi^{-1}(x) = y.$$

The full equivalence relation on (X, μ_p) is graphed by the set

$$\Phi = \{\phi_s\}_{s\in S}$$

where ϕ_s corresponds to the action by the generator s^{-1}. The cluster equivalence relation is graphed by

$$\Phi^{cl} = \{\phi_s^{cl}\}_{s\in S}$$

where ϕ_s^{cl} is the restriction of ϕ_s to the set of elements $x \in X$ such that the edge (e_Γ, s) lies in x. In the graphing $\mathcal{R}$ of E_Γ^{cl} on (X, μ_p), for every $x \in X$, $\mathcal{R}|[x]_{E_\Gamma^{cl}}$ is isomorphic to the cluster containing e_Γ in x. Since for almost every $x \in X$, every infinite cluster has infinitely many ends, the graphing $\mathcal{R}$ has infinitely many ends almost everywhere as well.

For a non-amenable group Γ with Cayley graph $\mathcal{G}$, E_Γ^{cl} is non-amenable exactly when $p > p_c(\mathcal{G})$. If there are infinitely many infinite clusters μ_p-a.e., then the resulting equivalence relation has infinitely many ends, which implies non-amenability. If there is just one infinite cluster, then R_Γ coincides with R_Γ^{cl} on the set $U \subseteq X$ of elements that have an infinite cluster at the identity, which has μ_p-positive measure. Conversely, the cluster equivalence relation is μ_p-almost everywhere finite for $p < p_c(\mathcal{G})$ and μ_p-hyperfinite for $p = p_c(\mathcal{G})$; it was first established that at $p_c(\mathcal{G})$, the clusters are μ_{p_c}-almost everywhere finite in [BLPS99].

We can now look at the infinite clusters in (X, μ_p) in terms of the associated cluster equivalence relation. Specifically, we may conclude the following:

Corollary 5.4. *Let Γ be a finitely generated non-amenable group with a normal infinite amenable subgroup Δ. Let $\mathcal{G}$ be a Cayley graph of Γ with a non-unique percolation phase and $p \in (p_c, p_u)$. Then for μ_p-almost every $x \in \{0,1\}^E$, every infinite cluster has only finitely many points in Δ.*

Remark After learning of this result by email, Lyons [Lyo] obtained Corollary 5.4 for amenable subgroups of finitely generated non-amenable groups without the assumption of normality. His argument is also more direct than ours.

In contrast, a cluster's interaction with Γ in $\Gamma \times \mathbb{Z}$ is very different even when the cluster does not have infinitely many ends, as witnessed by the following:

Corollary 5.5. *Consider $\mathcal{G} = (V, E)$ be a locally finite Cayley graph of $\Gamma \times \mathbb{Z}$ where Γ is a non-amenable group. Let $p > p_c(\mathcal{G})$. Then for μ_p-almost every $x \in \{0,1\}^E$, there is some $n \in \mathbb{N}$ such that*

$$x \cap (\Gamma \times [-n, n])$$

contains an infinite cluster in the induced subgraph structure.

Proof. We establish this on a subset of (X, μ_p) of positive measure and conclude the result by ergodicity of the cluster equivalence relation when restricted to the subset of X that has an infinite cluster at the identity. We show that if this is not the case, then $E_\Gamma \cap F$ is μ_p-hyperfinite, where $F = E_\Gamma^{cl}$ is the cluster equivalence relation and E_Γ is the orbit equivalence relation arising from the canonical action on Γ by shift. We define an increasing sequence $\{E_n\}_{n \in \mathbb{N}}$ of equivalence relations as follows:

xE_ny if and only if

(1) $\exists \gamma \in \Gamma, (\gamma, 0)^{-1} \cdot x = y$;
(2) $(\gamma, 0)$ is in the connected component of $e_{\Gamma \times \mathbb{Z}}$ in $x \cap (\Gamma \times [-n, n])$.

It suffices to check that each E_n is an equivalence relation since it is evident that $E_n \subseteq E_\Gamma \cap F$ and $\bigcup_{n \in \mathbb{N}} E_n = E_\Gamma \cap F$.

Indeed, suppose that for some $n \in \mathbb{N}$ and $x, y, z \in X$, we have xE_ny and yE_nz. Then let $\gamma_1, \gamma_2 \in \Gamma$ witness the first condition above for xE_ny and yE_nz, respectively. Then

$$z = (\gamma_2, 0)^{-1} \cdot y = (\gamma_2, 0)^{-1}(\gamma_1, 0)^{-1} \cdot x = (\gamma_1\gamma_2, 0)^{-1} \cdot x.$$

There is a path p_1 from $(e_\Gamma, 0)$ to $(\gamma_1, 0)$ in $x \cap (\Gamma \times [-n, n])$ and a path p_2 from $(e_\Gamma, 0)$ to $(\gamma_2, 0)$ in $((\gamma_1, 0)^{-1} \cdot x) \cap (\Gamma \times [-n, n])$. As a result, $(\gamma_1, 0) \cdot p_2$ from $(\gamma_1, 0)$ to $(\gamma_1\gamma_2, 0)$ in $x \cap (\Gamma \times [-n, n])$. As a result, the concatenation of p_1 and $(\gamma_1, 0)p_2$ witnesses the second condition for x and z and we may conclude that xE_nz.

□

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