Measure Theory, 620-411
Partial Solutions for Homework One

Please note, I am not attempting to give full well worked out solutions for every single problem, but rather illustrate the key ideas. Admittedly I did not want to spend too much time writing up, but it is also true that seeing the crucial points in a telegraphic style may be more helpful than a ponderous essay type format with every single bell rung, bow tied, t crossed, and i dotted.

I will skip the questions which gave less trouble.

I marked Qs 2, 4, and 5 out of 5, and the multi part questions (1 and 3) were 3 points for each part. That comes to a total of 39, so everyone got a point for putting their name on the front.

Q1: *Let Σ be a σ-algebra on a set X. Let $f, g : X \to \mathbb{R}$ be measurable with respect to Σ.*

(i) Show that

$$x \mapsto |f(x)|$$

is a measurable (with respect to Σ) function.

The open intervals of the form (a, ∞) generate the Borel sets as a σ-algebra, thus it suffices to show that for a in $\mathbb{R}$ and $U = (a, \infty)$ we have $|f|^1(U) \in \Sigma$. Now it depends on the sign of a. If $a \leq 0$, then the pullback is clearly all of X. On the other hand, if $a > 0$, then the pullback equals

$$f^{-1}[V]$$

for $V = (-\infty, -a) \cup (a, \infty)$.

(ii) *Show that*

$$x \mapsto f(x) + g(x)$$

is a measurable (with respect to Σ) function.

Again, suffices to show that the set

$$\{x : f(x) + g(x) > a\}$$

is in Σ for some $a \in \mathbb{R}$. This in turn equals

$$\bigcup_{q,r \in \mathbb{Q}, q+r>a} \{x : f(x) > q, g(x) > r\}.$$

Each such set $\{x : f(x) > q, g(x) > r\}$ equals the intersection of $\{x : f(x) > q\}$ and $\{x : g(x) > r\}$, both of which are in Σ by the assumptions on f and g.

Q3: *Let*

$$\prod_{\mathbb{N}} \{H, T\}$$

be the collection of all functions $f : \mathbb{N} \to \{H, T\}$ equipped with the product topology (i.e. equip $\{H, T\}$ with the discrete topology and take the resulting product topology on $\prod_{\mathbb{N}} \{H, T\}$). For $\vec{i} = i_1, i_2, ..., i_N$ distinct elements of $\mathbb{N}$ and $\vec{S} = S_1, S_2, ..., S_N \in \{H, T\}$ let

$$A_{\vec{i},\vec{S}} = \{f \in \prod_{\mathbb{N}} \{H, T\} : f(i_1) = S_1, f(i_2) = S_2, ..., f(i_N) = S_N\}.$$

(i) Let Σ_0 be the collection of all finite unions of the sets of the form $A_{\vec{i},\vec{S}}$. Show that Σ_0 is an algebra. (i.e. closed under finite unions, intersections, and complements).

Not many problems here, though some people missed out basic things like showing closure under complements.

(ii) Define

$$\mu_0 : \{A_{\vec{i},\vec{S}} : N \in \mathbb{N}, \vec{i} = i_1, i_2, ..., i_N \text{ distinct}, \ \vec{S} = S_1, S_2, ..., S_N \in \{H, T\}\} \to \mathbb{R}^{\geq 0}$$

by

$$\mu_0(A_{\vec{i},\vec{S}}) = 2^{-N},$$

where $\vec{i} = i_1, i_2, ..., i_N$ *distinct. Show that* μ_0 *extends to a function on* Σ_0 *which is* σ*-additive on its domain.*

This seemed to create a *lot* of difficulties, and in many cases it seemed to me people were simply not answering the question – for instance, showing finite additivity, but not σ-additivity on its domain. A number of people seemed to be almost trying to argue that there is no infinite sequence of non-empty disjoint sets in Σ_0 – which is, of course, plainly false.

There are two main steps: First finite additivity, then use compactness to extend it to the general case.

Step one: It is helpful here to introduce a couple more definitions. For a set of the form $A = A_{\vec{i},\vec{S}}$, we let $I(A)$ be the largest i which appears on the $\vec{i}$ sequence. This is, as it were, something like the "support" of A – the largest coordinate needed to complete the definition of the set. Now, for $n \in \mathbb{N}$, we let

$$X_n = \prod_{i \leq n} \{H, T\},$$

we let ν_n be the counting measure on X_n, and we let

$$\pi_n : \prod_{i \in \mathbb{N}} \{H, T\} \to X_n$$

be the obvious surjection obtained by restricting an f to the first n coordinates.

Note then that ν_n is the push forward of μ_0, in the sense that

$$\mu_0(\pi^{-1}[B]) = \mu_n(B),$$

when $B \subset X_n$.

Then given any disjoint $A_1, ...A_k \in \Sigma_0$, we can let N be the maximum value of the various $I(A_k)$'s. Then each A_k will equal some $\pi^{-1}[B_k]$, for some choice of $B_k \subset X_N$. Now the finite additivity question reduces to the case for the counting measure.

Step two: The general case for σ-additivity can now be reduced to step one. Suppose $A, A_0, A_1, A_2, ..., A_i, ...$ are all in Σ_0 and

$$A = \dot{\bigcup}_{n \in \mathbb{N}} A_n.$$

Then in the product topology all these sets are clopen (i.e. closed and open). In particular A is compact, since it is a closed subset of a compact space, and each A_n is open. Hence we can find a finite N such that

$$A = \bigcup_{n \leq N} A_n.$$

(iii) Show that μ_0 *extends to a measure* μ *on the Borel subsets of* $\prod_{\mathbb{N}} \{H, T\}$.

Given the work in (ii), this amounts to quoting the appropriate extension theorem from the course.

(iv) *At each* N *let* A_N *be the set of* $f \in \prod_{\mathbb{N}} \{H, T\}$

$$|\{n < N : f(n) = H\}| < \frac{N}{3}.$$

Show that $\mu(A_N) \to 0$ *as* $N \to \infty$.

Let us consider some large N, and in order to simplify the calculations assume N can be written in the form

$$N = 6M.$$

Now if $f \in A_N$ then for some $k < M$, we have exactly k places where f chooses the value H. There are

$$C(6M, k) = \frac{6M!}{(6M-k)!k!}$$

ways in which this can be done. Let us take for granted that $C(6M, k)$ becomes smaller as k moves away from $3M$. Thus the total of number of ways in which f can fall into A_N (and here I am just as it were projecting to the counting measure on $\prod_{n \leq N}\{H, T\}$) is bounded by

$$3M \cdot \frac{6M!}{4M!2M!}.$$

On the other hand, the number of ways in which f can assume exactly M many H's is equal to

$$C(6M, 3M) = \frac{6M!}{3M!3M!}.$$

Dividing the equation before last by this one, we obtain an upper estimate on the probability of $f \in A_N$ with

$$3M \cdot \frac{3M \cdot (3M-1) \cdot ... \cdot (2M+1)}{4M \cdot (4M-1) \cdot ...(3M+1)}.$$

This in turn bounded by

$$3M \cdot (\frac{3}{4})^{\frac{M}{4}}.$$

So now the problem (under the simplifying assumption N is divisible by 6) reduces to seeing that for $c > 0$ we have that

$$\frac{x}{e^{cx}}$$

approaches zero as x approaches infinity. At this stage I would probably be willing to accept the simple assertion that this is something we all learn in undergraduate calculus. One way to see it, though, is to appeal to the power series expansion of e^x and note that it is bounded by

$$\frac{x}{1 + cx + 1/2c^2x^2 \cdots} < \frac{x}{1/2c^2x^2} = \frac{2}{c^2x},$$

which clearly goes to zero as $x \to \infty$.

For the case of general N, we can write it in the form of $N = 6M + k$, some $k < 6$. Then if an arbitrary f is in A_N, it will in particular be in A_{6M} and there are at most $2^k \leq 2^5$ possibilities for its values on the coordinates $\{6M+1, 6M+2, ..., 6M+k\}$. This gives

$$\mu(A_N) \leq 2^5\mu(A_{6M}),$$

and we have reduced the general case to the specific one above.

I do accept that Q3 is rather involved, especially in light of this last calculation. I did want to draw some very brief connection with the "law of large numbers", which roughly states that as we run larger and larger samples of independent experiments and take the average, that average value almost surely converges to the expected value.

Q4: *Let $A \subset \mathbb{R}$ be Lebesgue measurable. Show that $m(A)$ is the supremum of*

$$\{m(K) : K \subset A, K \text{ compact}\}.$$

It seemed to me that there was a *little bit* of fudging in *some* of the answers you gave me, such as quoting results we had proved in class for Borel sets but not general Lebesgue measurable sets.

First of all, we can assume that $A \subset [-N, N]$ some N. Then using

$$m^*([-N, N]) = m^*(A) + m^*([-N, N] \cap A^c)$$

we can find some open $O \supset [-N, N] \cap A^c$ such that

$$m(A) + m(O) < 2N + \epsilon.$$

Then it follows that $m([-N, N] \setminus O) > m(A) - \epsilon$.

Q5: *Suppose $B \subset \mathbb{R}$ and there are Borel sets A_1, A_2 with $A_1 \subset B \subset A_2$ and $m(A_2 \setminus A_1) = 0$. Show that B is Lebesgue measurable.*

For $A \subset \mathbb{R}$ we have

$$m^*(A) = m^*(A_2 \cap A) + m^*(A_2^c \cap A) = m^*(A_2 \cap A) + m^*(A_1^c \cap A) - m^*(A_2 \cap A_1^c \cap A),$$

by the Lebesgue measurability of A_1^c and A_2. Now using $m^*(A_2 \setminus A_1) = 0$ and the monotonicity of m^* we obtain that this equals by

$$m^*(A_2 \cap A) + m^*(A_1^c \cap A) \geq m^*(B \cap A) + m^*(B^c \cap A).$$