Borel Equivalence Relations:
Dichotomy Theorems and Structure

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Definition A space is *Polish* if it is separable and admits a complete metric.

We then say that the *Borel sets* are those appearing in the smallest σ-algebra containing the open sets.

A set X equipped with a σ-algebra is said to be a *standard Borel space* if there is some choice of a Polish topology giving rise to that σ-algebra as its collection of Borel sets.

A function between two Polish spaces,

$$f : X \to Y,$$

is said to be *Borel* if for any Borel $B \subset Y$ the pullback $f^{-1}[B]$ is Borel.

We have gone through a number of examples in the first two talks. There is a sense in which Polish spaces are ubiquitous.

The notion of standard Borel space is slightly more subtle.

However it turns out that there are many examples of standard Borel spaces which possess a *canonical* Borel structure, but no *canonical* Polish topology.[1]

Theorem 1.1 *(Classical) If X is a Polish space and $B \subset X$ is a Borel set, then B (equipped in the σ-algebra of Borel subsets from the point of view of X) is standard Borel.*

Theorem 1.2 *(Classical; the "perfect set theorem") If X is a Polish space and $B \subset X$ is a Borel set, then exactly one of:*

1. *B is countable; or*
2. *B contains a homeomorphic copy of Cantor space, $2^{\mathbb{N}}$ (and hence has size $2^{\aleph_0}$).*

[1] Indeed, since we are mostly only considering Polish spaces up to questions of Borel structure, it is natural to discount the specifics of the Polish topology involved.

Theorem 1.3 *(Classical) If X is a standard Borel space, then the cardinality of X is one of* $\{1, 2, 3, ..., \aleph_0, 2^{\aleph_0}\}$.

Moreover!

Theorem 1.4 *(Classical) Any two standard Borel spaces of the same cardinality are Borel isomorphic.*

Here we say that X and Y are Borel isomorphic if there is a Borel bijection

$$f : X \to Y$$

whose inverse is Borel.[2]

Thus, as sets equipped with their σ-algebras they are isomorphic.

There is a similar theorem for quotients of the form X/E, E a Borel equivalence relation.

[2] In fact it is a classical theorem that any Borel bijection must have a Borel inverse

2 The analogues for Borel equivalence relations

Definition If X is a standard Borel space, an equivalence relation E on X is *Borel* if it appears in the σ-algebra on $X \times X$ generated by the rectangles $A \times B$ for A and B Borel subsets of X.

Theorem 2.1 *(Silver, 1980) Let X is a standard Borel space and assume E is a Borel equivalence relation on X. Then the cardinality of X/E is one of*

$$\{1, 2, 3, ..., \aleph_0, 2^{\aleph_0}\}.$$

However here there is no *moreover.*

In terms of Borel structure, and the situation when X/E is uncountable, there are vastly many possibilities at the level of Borel structure.

Definition Given equivalence relations E and F on standard Borel X and Y we say that E is Borel reducible to F, written

$$E \leq_B F,$$

if there is a Borel function

$$f : X \to Y$$

such that

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

We say that the *Borel cardinality* X/E *is less than the Borel cardinality of* Y/F, written

$$E <_B F,$$

if there is a Borel reduction of E to F but no Borel reduction of F to E.

In the language Borel reducibility, there is a sharper version of Silver's theorem, which he also proved without describing it in these terms.

Theorem 2.2 *(Silver) Let E be a Borel equivalence relation on a standard Borel space. Then exactly one of:*

1. $E \leq_B \mathrm{id}(\mathbb{N})$*; or*

2. $\mathrm{id}(\mathbb{R}) \leq_B E$*.*

One of the major events in the prehistory of the subject is Leo Harrington's alternate and far shorter proof of Silver's result using a technology called *Gandy-Harrington forcing*.

Building on this technology with the combinatorics of an earlier argument due to Ed Effros, the whole field of Borel equivalence relations was framed by the landmark theorem of Harrington, Kechris, and Louveau.

Recall that E_0 is the equivalence relation of eventual agreement on infinite binary sequences.

Theorem 2.3 *(Harrington, Kechris, Louveau, 1990) Let E be a Borel equivalence relation on a standard Borel space. Then exactly one of:*

1. $E \leq_B \mathrm{id}(\mathbb{R})$; or

2. $E_0 \leq_B E$.

This raised the fledgling hope that we might be able to provide a kind of structure theorem for the Borel equivalence relations under $\leq_B$, but before recounting this part of the tale I wish to describe the analogies which exist in the theory of $L(\mathbb{R})$ cardinality.

3 Cardinality in $L(\mathbb{R})$

Definition $L(\mathbb{R})$ is the smallest model of ZF containing the reals and the ordinals.

Although this quick formulation finesses out of the need to provide any set theoretical formalities, it rather disguises the true nature of this inner model.

It turns out that $L(\mathbb{R})$ can be defined by simply closing the reals under certain kinds of highly "constructive" operations carried out through transfinite length along the ordinals. It should possibly be thought of as the collection of sets which can be defined "internally" or "primitively" from the reals and the ordinals.

In this talk I want to think of it as a class inner model which contains anything one might think of as being a necessary consequence of the existence of the reals.

It also turns out that ZFC is incapable of deciding even the most basic questions about the theory and structure of $L(\mathbb{R})$.

On the other hand, if $L(\mathbb{R})$ satisfies AD, or the "Axiom of Determinancy" then almost all those ambiguities are resolved.

Following work of work of Martin, Steel, Woodin, and others, we now know that any reasonably large "large cardinal assumption" implies $L(\mathbb{R}) \models$ AD.

This along with the fact that $L(\mathbb{R}) \models$ AD has many regularity properties displayed by the Borel sets (such as the perfect set theorem for arbitrary sets in standard Borel spaces, all sets of reals Lebesgue measurable) has convinced many set theorists, though not all, that this is the right assumption under which to explore its structure.

I am *not* going to ask the audience to necessarily accept this perspective. I am simply going to examine the cardinality theory of $L(\mathbb{R})$ as a kind of idealization of the theory of Borel cardinality.

From now on in this part I will assume

$$L(\mathbb{R}) \models \mathrm{AD}.$$

Definition For A and B in $L(\mathbb{R})$, we say that the $L(\mathbb{R})$ *cardinality of* A *is less than or equal to the* $L(\mathbb{R})$ *cardinality of* B, written

$$|A|_{L(\mathbb{R})} \leq |B|_{L(\mathbb{R})},$$

if there is an injection in $L(\mathbb{R})$ from A to B. Similarly

$$|A|_{L(\mathbb{R})} < |B|_{L(\mathbb{R})},$$

if there is an injection in $L(\mathbb{R})$ from A to B but not from B to A.

Since the axiom of choice fails inside $L(\mathbb{R})$, there is no reason to imagine that the $L(\mathbb{R})$ cardinals will be linearly ordered, and in fact there *are* incomparable cardinals inside $L(\mathbb{R})$.

It turns out that the theory of $L(\mathbb{R})$ cardinality simulates and extends the theory of Borel cardinality.

In every significant case, the proof that

$$E <_B F$$

has also given a proof that

$$|X/E|_{L(\mathbb{R})} < |Y/F|_{L(\mathbb{R})}.$$

This is partially explained by:

Fact 3.1 *For E and F Borel equivalence relations one has*

$$E \leq_{L(\mathbb{R})} F$$

if and only if

$$|\mathbb{R}/E|_{L(\mathbb{R})} \leq |\mathbb{R}/F|_{L(\mathbb{R})}.$$

The two dichotomy theorems for Borel equivalence relations allow a kind of extension to the cardinality theory of $L(\mathbb{R})$.

Theorem 3.2 *(Woodin) Let* $A \in L(\mathbb{R})$*. Then exactly one of the following two things must happen:*

1. $|A|_{L(\mathbb{R})} \leq |\alpha|_{L(\mathbb{R})}$*, some ordinal* α*; or*

2. $|\mathbb{R}|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$.

Theorem 3.3 *(Hjorth) Let* $A \in L(\mathbb{R})$*. Then exactly one of the following two things must happen:*

1. $|A|_{L(\mathbb{R})} \leq |\mathcal{P}(\alpha)|_{L(\mathbb{R})}$*, some ordinal* α*; or*

2. $|\mathcal{P}(\omega)/\mathrm{Fin}|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$.

Here $|\mathcal{P}(\omega)/\mathrm{Fin}|_{L(\mathbb{R})} = |2^{\mathbb{N}}/E_0|_{L(\mathbb{R})}$, thus providing an analogy with Harrington-Kechris-Louveau.

4 Further structure

Definition Let E_1 be the equivalence relation of eventual agreement on $\mathbb{R}^{\mathbb{N}}$. For $\vec{x}, \vec{y} \in (2^{\mathbb{N}})^{\mathbb{N}}$, set $\vec{x}(E_0)^{\mathbb{N}}\vec{y}$ if at every coordinate $x_n E_0 y_n$.

Theorem 4.1 *(Kechris, Louveau) Assume*

$$E \leq_B E_1.$$

Then exactly one of:

1. $E \leq_B E_0$; *or*

2. $E_1 \leq_B E$.

Theorem 4.2 *(Hjorth, Kechris) Assume*

$$E \leq_B (E_0)^{\mathbb{N}}.$$

Then exactly one of:

1. $E \leq_B E_0$; *or*

2. $(E_0)^{\mathbb{N}} \leq_B E$.

Admittedly these are far more local in nature.

These are the only immediate successors to E_0 which we have established.

There is an entire spectrum of examples, constructed by Ilijas Farah using ideas from Banach space theory, for which it seems natural to suppose they must be minimal above E_0.

However this remains open, due to problems in the theory of countable Borel equivalence relations which appear unattainable using current techniques.

Moreover Alexander Kechris and Alain Louveau have shown that there is a sense in which there are no more global dichotomy theorems after Harrington, Kechris, Louveau.

5 Anti-structure

Theorem 5.1 *(Louveau, Veličković)* *There are continuum many many $\leq_B$ incomparable Borel equivalence relations.*[3]

In fact we can embed $\mathcal{P}(\mathbb{N})$ into $\leq_B$:

There is an assignment

$$S \mapsto E_S$$

of Borel equivalence relations to subsets of $\mathbb{N}$ such that for all $S, T \subset \mathbb{N}$ we have that $T \setminus S$ is finite if and only if

$$E_T \leq_B E_S.$$

Thus there is nothing like the kind of structure for Borel cardinality that one finds with the Wadge degrees.

[3] This first part may have been proved earlier by Hugh Woodin in unpublished work.

Theorem 5.2 *(Kechris, Louveau) There is no Borel $E >_B E_0$ with the property that for all other Borel F we always have one of:*

1. $F \leq_B E$; or

2. $E \leq_B F$.

Two key facts: First of all, Kechris and Louveau showed that E_1 is not Borel reducible to any E_G arising as a result of a continuous Polish group action[4], and secondly Leo Harrington showed that the Borel E_G's of this form are unbounded with respect to Borel reducibility:

Theorem 5.3 *(Harrington) There is a collection $\{E_\alpha : \alpha \in \omega\}$ of Borel equivalence relations such:*

1. *Each E_α arises as a result of the continuous Polish group action on a Polish space;*
2. *For any Borel F there will be some α with E_α* not *Borel reducible to F.*

[4]A theoreom due to Howard Becker and Alexander Kechris theorem on changing topologies in the dynamical context shows that there is no basically no difference between equivalence relations induced by continuous actions and induced by Borel actions. *However* it is important that the responsible group be a Polish group – there are certain traces of rigidity for Polish groups, whereas Borel actions of Borel groups can induce any Borel equivalence relation one cares to name

To sketch a proof by contradiction of Kechris and Louveau's result, suppose E was a Borel equivalence relation with the property that for all Borel F we have one of:

1. $F \leq_B E$; or
2. $E \leq_B F$.

Referring back to Harrington's theorem, there will be some α with E_α not Borel reducible to E.

Thus since 1 fails for $F = E_\alpha$ we must have $E <_B E_\alpha$

But E_1 is not Borel reducible to any Polish group action, and hence using the same reasoning we must have $E <_B E_1$.

Which by the Kechris-Louveau dichotomy theorem yields $E \leq_B E_0$.

However this proof prompts the following response:

Question Let E be a Borel equivalence relation. Must we have one of the following:

1. $E \leq_B E_G$ some E_G induced by the continuous action of a Polish group on a Polish space; or
2. $E_1 \leq_B E$?

In other words, is E_1 the *only* obstruction to "classification" or "reduction" to a Polish group action?

At present this is wide open.

The question has, however, been positively answered by Slawomir Solecki in many special cases. In particular, his penetrating structure theorem for Polishable ideals proves it for equivalence relations on $2^{\mathbb{N}}$ arising as the coset equivalence relation of some Borel ideal.

6 Dichotomy theorems for classification by countable structures

Definition An equivalence relation E on a Polish space X is *classifiable by countable structures* if there is a countable language $\mathcal{L}$ and a Borel function

$$f : X \to \mathrm{Mod}(\mathcal{L})$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow f(x_1) \cong f(x_2).$$

This notion of classifiability has been subject to close scrutiny, in part since it is so natural from the perspective of a logician.[5]

It might also provide a template of what we could hope to achieve with other notions of classifiability, where some kind of structure theorems can be proved without appeal to a Harrington, Kechris, Louveau type dichotomy theorem.

[5] In fact a Borel equivalence relation E will be $L(\mathbb{R})$ classifiable by countable structures if and only if $|X/E|_{L(\mathbb{R})} \leq |HC|$ – classifiability in this sense amounting to reducible to the hereditarily countable sets

Theorem 6.1 *(Farah) There is a family of continuum many Borel equivalence relations, $(E_r)_{r\in\mathbb{R}}$, such that:*

1. *each E_r is induced by the continuous action of an abelian Polish group on a Polish space; and*
2. *no E_r is classifiable by countable structures;*
3. *for $r \neq s$ the equivalence relations are incomparable with respect to Borel reducibility;*
4. *if $E <_B E_r$, any r, then E is essentially countable, and hence classifiable by countable structures.*

This says that there is no single canonical obstruction to be classifiable by countable in the way we find E_0 as a canonical obstruction to smoothness.

Definition Let G be a Polish group acting continuously on a Polish space X. For V an open neighborhood of 1_G, U an open set containing x, we let

$$O(x, U, V),$$

the U-V-local orbit, be the set of all $\hat{x} \in [x]_G$ such that there is a finite sequence

$$(x_i)_{i \leq k} \subset U$$

such that

$$x_0 = x, \qquad x_k = \hat{x},$$

and each

$$x_{i+1} \in V \cdot x_i.$$

Definition Let G be a Polish group acting continuously on a Polish space X. The action is said to be *turbulent* if:

1. every orbit is dense; and
2. every orbit is meager; and
3. for $x \in X$, the local orbits of x are all somewhere dense.

Farah's theorem tells us we can not find even finitely many Borel equivalence relations which are canonical obstructions for classification by countable structures.

Theorem 6.2 *(Hjorth) Let G be a Polish group acting continuously on a Polish space X with induced orbit equivalence relation E_G^X. Assume E_G^X is Borel.*

Then exactly one of:

1. *E_G^X is classifiable by countable structures; or*
2. *G acts turbulently on some Polish space Y and*
$$E_G^Y \leq_B E_G^X.$$

There are in fact cases where one can rule out the existence of turbulent actions by a group, and thus show all the orbit equivalence relations induced by a certain Polish group must be classifiable by countable structures.

Theorem 6.3 *Let G be a Polish group acting continuously on a Polish space X with induced orbit equivalence relation E_G^X. Assume E_G^X is Borel.*

Then exactly one of:

1. *E_G^X is smooth; or*
2. *G acts continuously on a Polish space Y with all orbits dense and meager and*

$$E_G^Y \leq_B E_G^X.$$

Definition If G is a Polish group acting on a Polish space X, we call X *stormy* if for every nonempty open $V \subseteq G$ and $x \in X$ the map

$$V \to [x]_G$$

$$g \mapsto g \cdot x$$

is not an open map.

In a manner parallel to the theory of turbulence stormy provides *the* obstruction for being essentially countable.

7 The wish list

Question Let E be a Borel equivalence relation.

Must we have one of:

1. $E \leq_B E_G$ for some E_G arising as the orbit equivalence relation of a Polish group acting continuously on a Polish space; or
2. $E_1 \leq_B E$?

More generally, if we could establish that there is some analysis of when an equivalence relation is Borel reducible to a Polish group action, then we could lever the theorems regarding turbulence and stormy actions to gain a general understanding of when a Borel equivalence relation admits classification by countable structures or is essentially countable.

Question Let E_G arise from the continuous action of an *abelian* Polish group on a Polish space. Let $E \leq_B E_G$ be a Borel equivalence relation with countable classes.

Must we then have $E \leq_B E_0$?

If so, then Farah's earlier examples would obtain continuum many immediate successors to E_0 in the $\leq_B$ ordering.

Other work of Farah would obtain Borel equivalence relations which are above E_0 but have no immediate successor to E_0 below.

Is there a kind of generalized dichotomy theorem for hyperfiniteness?

Most optimistically:

Question Let E be a countable Borel equivalence relation. Must we have either:

1. $E \leq_B E_0$; or
2. there is a free measure preserving action of $\mathbb{F}_2$ on a standard Borel probability space such that $E_{\mathbb{F}_2} \leq_B E$?

It *is* known that no such $E_{\mathbb{F}_2}$ is Borel reducible to E_0.

This seems wildly optimistic at present, and perhaps it would be less rash to ask it only in the case that E is treeable, but it would in particular have as one of its consequences a positive answer to the following:

Question Let G be a countable *amenable*[6] group. Suppose G acts in a Borel manner on a standard Borel space X.

Must we have $E_G \leq_B E_0$?

The closest result to this is given by a startlingly original combinatorial argument due to Su Gao and Steve Jackson who establish a positive answer in the case G is abelian.

There are no known techniques, or even hints at ideas, which could provide a counterexample to the above question.

[6] *Amenability* can be characterized as the statement that for all $F \subset G$ finite, $\epsilon > 0$, there is some $A \subset G$ finite with

$$\frac{|A \Delta g \cdot A|}{|A|}$$

all $g \in F$.

All known proofs that an equivalence relation is not reducible to E_0 rely on measure theory, and it follows from Connes, Feldman, Weiss that any such E_G must be Borel reducible to E_0 on some conull set with respect to any Borel probability measure.

In fact:

Question Let E be a countable Borel equivalence relation. Are measure theoretic reasons the *only obstruction* to being Borel reducible to E_0?

For instance, if E is countable and not Borel reducible to E_0, must it be the case that there is a Borel probability measure μ such that $E|_A$ is not Borel reducible to E_0 on any conull A?

Any counterexample would require the development of fundamentally new ideas about how to prove some equivalence relations are not $\leq_B$ E_0.